

ALGEBRAIC GEOMETRY AND
PROJECTIVE DIFFERENTIAL GEOMETRY
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Preface.

Please note: This is a preliminary version, the final version will be published in book form by Seoul National University. Please send comments, corrections and questions to me at jml@picard.ups-tlse.fr. I am particularly interested in historical comments, as I have been unable to trace the original authorship of several results (those marked with a ??).

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This is an expanded and updated version of a lecture series I gave at Seoul National University in September 1997. The series was focus on Zak's theorem on Severi varieties and my differential-geometric proof of the theorem at the request of my hosts. A French language version of the introduction served as the text for my *habilitation à diriger des recherches* (November 1997).

These notes are written with two different audiences in mind: graduate students and algebraic geometers not familiar with infinitesimal methods.

For the graduate students, I have included exercises and open questions to work on. For the algebraic geometers not familiar with infinitesimal methods, I have attempted to relate the techniques used here with standard methods in algebraic geometry.

These notes are in some sense an update to the paper [GH].

I do not discuss hyperdeterminants, Geometric Invariant Theory, Variation of Hodge Structure, web geometry, theta divisors and other related topics, although I would have liked to. I strongly encourage graduate students to study the connections with some of these other topics.

I would like to thank Jun-Muk Hwang and Seoul National University.

§0. INTRODUCTION

Let $X^n \subset \mathbb{CP}^{n+a} = \mathbb{P}V$ be a variety ($V = \mathbb{C}^{n+a+1}$). In these notes I discuss the geometry of X as an algebraic variety, the local projective differential geometry of the smooth points of X , and most importantly, the relations between the two.

In the spirit of F. Klein, let's consider a property of a variety *geometric* if it is invariant under projective transformations (that is, the action of $PGL(V)$ on $\mathbb{P}V$). For example, two geometric properties of a variety are its *dimension* (the dimension of its tangent space at a smooth point) and its *degree* (the number of points of intersection with a general linear space of complementary dimension). The first property is intrinsic, the second extrinsic.

One way to measure the pathology of X is to construct auxilliary varieties from X , and to calculate the difference between the expected and actual dimensions of these auxilliary varieties. In these notes we will study such auxilliary varieties using modern techniques combined with infinitesimal methods developed by E. Cartan and others.

In what follows I give an example of an auxilliary variety, namely the *secant variety* of X and discuss its study. This example is typical and will serve as a model for the other cases.

0.1 Secant varieties.

Given two points $x, y \in \mathbb{P}V$, there exists a unique line \mathbb{P}_{xy}^1 containing them. Given a subvariety $X \subset \mathbb{P}V$, define the *secant variety* of X ,

$$\sigma(X) := \overline{\bigcup_{x,y \in X} \mathbb{P}_{xy}^1},$$

the closure of the union of all secant lines to X . There are $2n$ -dimensions of pairs of points on X and one parameter of points on each line, so one expects that $\dim \sigma(X) = 2n + 1$ if $2n + 1 \leq n + a$, or $\sigma(X) = \mathbb{P}^{n+a}$ if $2n + 1 > n + a$. If not, we say $\sigma(X)$ is *degenerate* and let $\delta_\sigma = 2n + 1 - \dim \sigma(X)$ denote the *secant defect* of X .

0.1.1 Example. Let $V = \mathbb{C}^{k+1} \otimes \mathbb{C}^{l+1}$ denote the space of $(k+1) \times (l+1)$ matrices. Let $X \subset \mathbb{P}V$ be the projectivization of the rank one matrices. $X \simeq \mathbb{P}^k \times \mathbb{P}^l$ because every rank one matrix is the product of a column vector with a row vector. $X = \text{Seg}(\mathbb{P}^k \times \mathbb{P}^l)$ is called the *Segre variety*. X is the zero locus of the two by two minors.

The sum of two rank one matrices has rank at most two, so $\sigma(\text{Seg}(\mathbb{P}^k \times \mathbb{P}^l)) = \mathbb{P}(\text{rank} \leq 2 \text{ matrices})$.

0.1.1.1 Exercise. $\dim \sigma(X) = 2(k+l) - 1$, so $\sigma(X)$ is degenerate, with $\delta_\sigma = 2$.

A general principle is that pathology should be rare if X is smooth and $\text{codim } X$ is relatively small (see §3). A theorem to this effect is the following:

0.1.2 Zak's theorem on linear normality [Z]. *If $X^n \subset \mathbb{P}^{n+a}$ is smooth, not contained in a hyperplane, and $a < \frac{n}{2} + 2$, then $\sigma(X) = \mathbb{P}^{n+a}$.*

(The name of this theorem is explained in §3.)

In addition, Zak (with Lazarsfeld) classified the varieties X in the borderline case of $a = \frac{n}{2} + 2$ and $\sigma(X)$ degenerate:

0.1.3 Zak's theorem on Severi varieties [Z], [LV]. *If $X^n \subset \mathbb{P}^{n+a}$ is smooth, not contained in a hyperplane, $a = \frac{n}{2} + 2$ and $\sigma(X) \neq \mathbb{P}^{n+a}$, then X is one of*

- i. Veronese $\mathbb{P}^2 \subset \mathbb{P}^5$
- ii. Segre $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$
- iii. Plücker embedded Grassmannian $G(\mathbb{C}^2, \mathbb{C}^6) \subset \mathbb{P}^{14}$
- iv. $E_6/P_1 \subset \mathbb{P}^{26}$.

These four varieties are called the *Severi varieties* after F. Severi who proved the theorem in the special case $n = 2$. They are described in §1 along with many other homogeneous varieties. Homogeneous varieties often provide examples of extremal pathologies so I discuss them in §2. In [L5] I give new proofs of Zak's theorems. The proofs have five steps, which will serve as a model for many of the questions discussed in these notes.

Step 1: Describe the pathology infinitesimally.

The condition imposed on the differential invariants of X when $\delta_\sigma > 0$ was determined by Terracini in 1913 [T]. The condition $\delta_\sigma > 1$ is essentially that the quadrics in the projective second fundamental form satisfy a polynomial (second fundamental forms will be defined shortly). See §9 for the precise condition. In particular, it is a closed condition. For other questions, such as in the study of complete intersections, this step can be quite involved, see §12 and 0.5.

Step 2: Analyze the infinitesimal condition.

Here one determines which systems of quadrics satisfy the Terracini condition. This type of question can be studied from several perspectives. To reprove Zak's theorems, I localize the problem yet again and use differential-geometric methods, see §10. In our study of dual varieties [IL], Ilic and I used methods from algebraic geometry at this stage; the study of vector bundles on projective space, see §7.

Step 3: Determine infinitesimal consequences of smoothness.

In this step one studies the additional conditions placed on the differential invariants of X because X is smooth (or almost smooth). These are usually open (genericity) conditions. Recovering global information from infinitesimal invariants is central to my research. See §4 and §7.

Step 4: Combine steps two and three.

In the case of degenerate secant varieties, one combines the open conditions implied by smoothness with the restrictions on the systems of quadrics arrived at in the second step. At this point Zak's theorem on linear normality follows immediately. In the Severi variety case, one is restricted to four possible second fundamental forms.

Step 5: Pass from infinitesimal to local (and hence global) geometry.

This step of passing from the infinitesimal geometry to the local geometry (and since one is in the analytic category, the result is a global) is made using the Cartan machinery of moving frames and exterior differential systems. This step can be viewed as the study of the deformation theory, or rigidity, of systems of quadrics, cubics, etc... Such results are discussed in §13.

I consider step 3 to be the most important, so let's begin there:

0.2 A principle relating global smoothness to local projective geometry (the third step).

In projective space, the global geometry restricts the local geometry. One can view these restrictions as consequences of the very defining property of the projective plane: that parallel lines meet at infinity, or, more generally, *in projective space, linear spaces (in fact arbitrary varieties) of complementary dimension must intersect*.

At each point x of a submanifold $X \subset \mathbb{P}V$, there is a unique *embedded tangent space*, that is, a unique linear space that best approximates X at x to first order. I denote the embedded tangent space by $\tilde{T}_x X$ to distinguish it from the intrinsic holomorphic tangent space, which I denote $T_x X$.

Consider the following two surfaces in affine space \mathbb{A}^3

$$(0.2.1) \quad \begin{array}{ccc} \text{hyperbola} & & \text{cylinder} \end{array}$$

Both the hyperbola and the cylinder are defined by a quadratic equation, and both are ruled by lines. Both can be completed to projective varieties by considering $\mathbb{A}^3 \subset \mathbb{P}^3$. When one completes the hyperbola, one obtains a smooth quadric surface in \mathbb{P}^3 . In contrast, completing the cylinder, one obtains a singular cone. The cylinder obtains a singularity because as one travels along one of its rulings, the embedded tangent space $\tilde{T}_x X$ is constant. This forces the rulings to crash into each other at infinity. In contrast, the embedded tangent space of the hyperbola rotates as one travels along a ruling and a singularity at infinity is thus avoided.

The contrast between these cases leads to the following principle:

0.2.2 Smoothness Principle [L6]. *In order for X to be smooth, its embedded tangent space must “move enough”.*

How much the tangent space needs to move will depend on $\dim X$ and $\text{codim } X$.

In order to make the smoothness principle precise, we need a way to measure “how much” $\tilde{T}_x X$ is moving. We will make such measurements using the *projective second fundamental form*.

0.3 The projective differential geometry of X .

Recall that in Euclidean geometry, the basic measure of how a submanifold of Euclidean space is bending (that is, moving away from its embedded tangent space to first order) is the Euclidean second fundamental form. In projective geometry, there is a *projective second fundamental form* that can be defined the same way as its Euclidean analogue:

0.3.1 The Gauss map γ and a definition of II via γ .

A natural way to keep track of the motion of $\tilde{T}_x X$ is the *Gauss map*

$$(0.3.1.1) \quad \begin{aligned} \gamma : X &\rightarrow \mathbb{G}(n, \mathbb{P}V) \\ x &\mapsto \tilde{T}_x X. \end{aligned}$$

where $\mathbb{G}(n, \mathbb{P}V)$ is the *Grassmannian* of \mathbb{P}^n 's in $\mathbb{P}V$. A \mathbb{P}^n in $\mathbb{P}V$ is equivalent to an $n+1$ plane passing through the origin in V . I use the notation $G(n+1, V) = \mathbb{G}(n, \mathbb{P}V)$ when I want to emphasize this second description.

To measure how $\tilde{T}_x X$ moves to first order, one calculates the derivative of γ :

$$(0.3.1.2) \quad \gamma_{*x} : T_x X \rightarrow T_{\hat{T}_x} G(n+1, V) = \hat{T}^* \otimes (V/\hat{T}) = \hat{T}^* \otimes N(-1).$$

Here I use the notation $N_x X = V/\hat{T} \otimes \hat{x}^*$ and $N(-1) = V/\hat{T}_x X$. If you are not familiar with such notations, there is little harm in thinking of both N and $N(-1)$ as the normal space to X at x . (If you are concerned, see (1.2).) Here and in what follows, I often omit reference to X and the base point x .

γ_{*x} is such that the kernel of the endomorphism $\gamma_*(v) : \hat{T} \rightarrow N(-1)$ contains $\hat{x} \subset \hat{T}$ for all $v \in T$. Thus γ_* factors to a map

$$(0.3.1.3) \quad \gamma'_* : T \rightarrow (\hat{T}/\hat{x})^* \otimes N(-1) = T^* \otimes N.$$

Furthermore, γ'_* is symmetric, essentially because the Gauss map is already the derivative of a map and mixed partials commute, (see (4.4) for a proof). It is called the *projective second fundamental form*, and denoted

$$(0.3.1.4) \quad II := \gamma'_* \in S^2 T^* \otimes N.$$

Roughly, II measures how X moves away from its tangent space at each point to first order. In other words, if one considers X as being mapped into $\mathbb{P}V$, II is the geometrically relevant second derivatives of the mapping.

Note that II is an *algebraic* object, and thus can be used in finite characteristic etc... as long as the usual precautions are taken.

0.3.2 A coordinate definition of II .

Let $x \in X$ be a smooth point. Choose local coordinates (x^1, \dots, x^{n+a}) around x such that $x = (0, \dots, 0)$ and $T_x X = \{\frac{\partial}{\partial x^\alpha}\}$, $1 \leq \alpha, \beta \leq n$ and $N_x^* X = \{dx^\mu\}$, $n+1 \leq \mu, \nu \leq n+a$. Write X locally as a graph $x^\mu = f^\mu(x^\alpha)$. Then, in these coordinates, the projective second fundamental form of X at x is:

$$(0.3.2.1) \quad II_{X,x} = \frac{\partial^2 f^\mu}{\partial x^\alpha \partial x^\beta} |_x dx^\alpha \circ dx^\beta \otimes \frac{\partial}{\partial x^\mu} \in S^2 T_x^* X \otimes N_x X.$$

0.3.3 Interpretations and measurements.

It is convenient to consider II as a map $II : N^* \rightarrow S^2 T^*$ (dual to the standard Euclidean perspective) and to set $|II| = \mathbb{P}II(N^*)$. One can think of $|II|$ as a linear family of quadric hypersurfaces in $\mathbb{P}T$.

$\mathbb{P}N_x^* X$ has the geometric interpretation as the space of hyperplanes tangent to X at x , i.e., the hyperplanes H such that $X \cap H$ is singular at x . II is essentially the map that (up to scale) sends a hyperplane to the quadratic part of the singularity of $X \cap H$ at x .

Step one for the secant variety problem leads to the following condition: if $a \leq n$, then $\delta_\sigma > 1$ implies that for all $v \in T$, there exists $q \in |II|$ such that $[v] \in q_{sing}$. See §8 for the degeneracy condition in the general case. The study of systems of quadrics satisfying this condition (the second step) is complicated, so I will wait until §10 to discuss it.

Returning to the general study of $|II|$, a natural question is:

How much of the geometry of X is determined by $|II|$ at a general point?

To investigate this question, we need to extract geometric information from $|II|$.

In projective geometry, unlike Euclidean geometry, one cannot measure how fast a submanifold is bending, but only whether or not it is bending.

For example, in projective geometry, one can measure if a line in the embedded tangent space \tilde{T} osculates to order two, i.e., if X appears to contain the line to second order. Let $\text{Base}|II| \subset \mathbb{P}T$ denote the variety of directions tangent to the lines that osculate to order two at x . That is,

$$(0.3.3.1) \quad \begin{aligned} \text{Base}|II| &= \mathbb{P}\{v \in T \mid II(v, v) = 0\} \\ &= \{[v] \in \mathbb{P}T \mid [v] \in q \forall q \in |II|\}. \end{aligned}$$

Given $H \in \mathbb{P}N_x^* X$, the quadric hypersurface $q_H \subset \mathbb{P}T_x X$ is the set of tangent directions not moving away from H to second order. If $c(t)$ is a curve lying in X with $c(0) = x$, then $[c'(0)] \in \text{Base}|II|$ if and only if $[c''(0)] \in \tilde{T}_x X$ (assuming $c''(0) \neq 0$).

A stronger condition on a tangent vector v is that the embedded tangent space does not move to first order in the direction of v . Let $\text{singloc}|II|$ denote the set of such directions. That is,

$$(0.3.3.2) \quad \begin{aligned} \text{singloc}|II| &:= \mathbb{P}\{v \in T \mid II(v, w) = 0 \forall w \in T\} \\ &= \{[v] \in \mathbb{P}T \mid [v] \in q_{sing} \forall q \in |II|\}. \end{aligned}$$

Note that the first equality implies that $\text{singloc}|II|$ is a linear subspace of $\mathbb{P}T$. If $c(t)$ is a curve lying in X with $c(0) = x$, then one can interpret the condition $c'(0) \in \text{singloc}|II|$ as follows: let $E_t := \tilde{T}_{c(t)} X$ denote the projective bundle defined over the curve $c(t)$. $[c'(0)] \in \text{singloc}|II|$ if and only if $E'_0 \subseteq \hat{T}_{c(0)} X$.

In terms of our invariants of the second fundamental form, directions tangent to the rulings of the cylinder are in $\text{singloc}|II|$. Directions tangent to the rulings of the hyperbola are in $\text{Base}|II|$, but not $\text{singloc}|II|$.

The study of projective second fundamental forms leads one back to algebraic geometry: the geometry of systems of quadrics. Sections §8 and §10 are dedicated to studying properties of systems of quadrics.

The smoothness principle is illustrated by the following theorem of Griffiths and Harris (??) that generalizes the example of the cylinder above:

0.3.3.3 Theorem, [GH]. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety. Let $x \in X$ be a general point. If $\text{singloc}|II|_x \neq \emptyset$ then X is singular.*

A more precise version of this theorem is given in §5.

The following theorem is another illustration of the smoothness principle. It states that the embedded tangent space must move away from *each* tangent hyperplane in a minimum number of directions:

0.3.3.4 Theorem, (special case of) rank restrictions, [L3]. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety. Let $b = \dim(X_{\text{sing}})$. (Set $b = -1$ if X is smooth.) Let $x \in X$ be a general point.*

1. For any quadric $q \in |II|_x$,

$$\dim(\text{Singloc } q) \leq 2(a-1) + (b+1).$$

2. For generic quadrics $q \in |II|_x$,

$$\dim(\text{Singloc } q) \leq a-1 + (b+1).$$

The rank restriction theorem is analogous to Bochner type formulae in differential geometry in which global considerations impose pointwise conditions on the curvature.

Varieties with the degeneracies that are discussed in these notes can be viewed as solutions to systems of partial differential equations. From this perspective, the rank restriction theorems have the effect of ruling out characteristic or degenerate initial data to initial value problems.

A long term goal is to prove stronger and higher order versions of the rank restriction theorem.

Before describing other problems and results, I will briefly motivate the method of calculation, the *moving frame*.

0.4 The moving frame.

Let $X \subset \mathbb{P}V$ be a variety and $x \in X$ a smooth point. Then $x \in X$ determines a flag

$$\hat{x} \subset \hat{T}_x X \subset V.$$

One way to study the geometry of X is to examine how this flag varies as we vary x . For example, the second fundamental form contains the information of how \hat{T} moves to first order in relation to the motion of \hat{x} . When studying particular geometric properties,

one often refines this flag. For example, in the study of varieties with degenerate secant varieties, it is convenient to also fix a vector $v \in \hat{T}/\hat{x}$, to obtain a flag

$$\hat{x} \subset \{\hat{x}, v\} \subset \hat{T} \subset V.$$

This flag admits further refinements, e.g., there is a subspace $\text{Ann}(v) \subset N_x X$ corresponding to the quadrics $q_H \in |II|$ annihilating v , and also a subspace $\text{singloc Ann}(v) \subset \hat{T}/\hat{x}$ consisting of all the vectors $v \in T$ annihilated by $\text{Ann}(v)$. Thus we obtain a refined flag:

$$\hat{x} \subset \{\hat{x}, v\} \subset \{\hat{x}, \text{singloc}(\text{Ann}(v))\} \subset \hat{T} \subset \{\hat{T}, \text{Ann}(v)\} \subset V.$$

One would like to keep track of the infinitesimal motions of all these spaces and their relations. Fortunately there is a method developed by G. Darboux, E. Cartan and others, the *moving frame*, exactly designed for this purpose. It keeps track of the relations between all the infinitesimal motions, and is designed in such a way that as further refinements of the flag are made, a minimal amount of additional work is necessary. From a geometrical perspective, it would be natural to begin on the manifold of complete flags of V . However a computational advantage is gained if one works on a slightly larger space, the space of bases, or *framings* of V , as this is a Lie group ($GL(V)$ in fact) and on a Lie group, derivatives can be calculated algebraically. Moving frames for subvarieties of projective space are discussed in §4.

In the remainder of this introduction, I describe additional problems in projective geometry and results I have obtained using the methods discussed above in their study. In the chapters I have attempted to give an overview of what is known in general regarding these questions.

0.5 Complete intersections.

The least pathological algebraic varieties are the smooth hypersurfaces. For example, the dimension is obvious and the degree is simply the degree of the single polynomial defining X . (Varieties are reduced and irreducible.)

A class of varieties that share many of the simple properties of hypersurfaces is the class of *complete intersections*.

0.5.1 Definition. A variety $X^n \subset \mathbb{P}^{n+a}$ is a *complete intersection* if the ideal of X , I_X , can be generated by a elements.

The following example shows how varieties that are not complete intersections can arise:

0.5.2 Example. In \mathbb{P}^3 , let Q_1, Q_2 be quadric hypersurfaces. Consider $X = Q_1 \cap Q_2$ (the common zero locus of two degree 2 homogeneous polynomials). X is an algebraic set of dimension one and degree four. If Q_1, Q_2 are reasonably general, then X is a curve of degree four:

(0.5.3)

But consider the following example: let (x^1, \dots, x^4) be linear coordinates on \mathbb{C}^4 , and let

$$\begin{aligned} Q_1 &= x^1 x^4 - x^2 x^3 \\ Q_2 &= (x^2)^2 - x^1 x^3 \end{aligned}$$

Then X is a curve of degree three plus a line:

(0.5.4)

$$C = [s^3, s^2 t, s t^2, t^3], [s, t] \in \mathbb{P}^1, l = [0, 0, u, v], [u, v] \in \mathbb{P}^1.$$

Varieties are irreducible, thus we need to get rid of one of these components. We pretend we understand lines, so we eliminate l by intersecting X with $Q_3 = (x^3)^2 - x^2 x^4$ to be left with the cubic curve. Degree $(C) = 3$, so C cannot be the intersection of two hypersurfaces. (If it were, it would have to be in a cubic hypersurface and a hyperplane, but C is not contained in any linear subspace.)

The difference between the topology of complete intersections and non-complete intersections has been studied extensively. I have attempted to understand how the projective differential geometries of the complete intersections and non-complete intersections differ.

If one compares the two pictures

(0.5.5)

and tries to understand the difference between them, one might say that the non-complete intersection “bends less”, or that its tangent space “moves less” than the complete intersection. (This idea can be made precise if one adds a Kähler metric and is willing to integrate. Since we will work locally, this is not what we will do.) This idea is central to what follows, so I record it informally:

0.5.6 Complete intersection principle. *If X “bends enough”, then X will be a complete intersection.*

The determination of how much is “enough” will be based on information about the degrees of hypersurfaces containing X . In our example, the cubic curve “bends less” than a complete intersection of quadrics would.

A precise explanation of the phrase “bends enough” is given below. For now, consider the curve in affine space $y = x^3$.

(0.5.7)

At the origin, the embedded tangent space “moves less” than at other points on the curve in the sense that the curve *osculates* to its tangent line to order two at the origin (versus order one for all other points). (0.5.7) motivates one to study bending. However, the actual type of bending we will study will concern whether or not there is osculation to orders higher than expected (in dimensions greater than one).

Let’s now discuss what one means by orders of osculation “higher than expected”. To decide what order of osculation is expected, I determine *a priori* information about osculating hypersurfaces. The following results are of interest, independently of the study of complete intersections.

If $x \in X$ is a smooth point, then there is always an $(a-1)$ -dimensional space of hyperplanes (degree one hypersurfaces) tangent (osculating to order one) at x . The following proposition generalizes the case of hyperplanes:

0.5.8 Proposition [L4,3.16]. *Let $X^n \subseteq \mathbb{P}^{n+a}$ be a variety and let $x \in X_{sm}$. For all $p \leq d$,*

$$\begin{aligned} \dim & \left\{ \begin{array}{l} (\text{not necessarily irreducible}) \text{ hypersurfaces} \\ \text{of degree } d \text{ osculating to order } p \text{ at } x \end{array} \right\} \\ &= \binom{n+a+d}{d} - \binom{n+p}{p}. \end{aligned}$$

For $k > d$, the dimensions of the spaces of osculating hypersurfaces depend on the geometry of X . One might think that the pattern would continue, so that if a is relatively small one would expect that generically no hypersurfaces would osculate to order $d+1$. However, at higher orders, a new phenomena occurs because of hypersurfaces that are singular at x . Independent of X , for $d+1 \leq k \leq 2d-1$, there are *lower* bounds on the dimensions of the space of hypersurfaces of degree d osculating to order k at x . For example:

0.5.9 Proposition [L4,3.17]. *Let $X^n \subseteq \mathbb{P}^{n+a}$ be a variety, and let $x \in X_{sm}$.*

$$\dim \left\{ \begin{array}{l} (\text{not necessarily irreducible}) \text{ hypersurfaces of} \\ \text{degree } d \text{ osculating to order } 2d-1 \text{ at } x \end{array} \right\} \geq \binom{a+d-1}{d} - 1.$$

One possible definition of “bending less” would be that the lower bounds for $d+1 \leq k \leq 2d$ are attained (modulo the contribution of $I_{d-1} \circ V^*$).

Singular osculating hypersurfaces are the key to understanding the projective geometry of non-complete intersections. To explain why, for notational simplicity, assume X is the intersection of hypersurfaces of degree d (see §12 for the general case).

0.5.10 Proposition [L4, 1.1]. *Let $X \subset \mathbb{P}V$ be a variety such that $I_X = (I_d)$ (i.e. I_X is generated by I_d) and $I_{d-1} = (0)$. Then the following are equivalent:*

1. *X is a complete intersection.*
2. *Every hypersurface of degree d containing X is smooth at all $x \in X_{sm}$.*
3. *Let $x \in X_{sm}$. Every hypersurface of degree d containing X is smooth at x .*

Thus if X is a complete intersection, there can be no hypersurfaces singular at any $x \in X_{sm}$ containing X that osculate to order infinity.

If X is not a hypersurface, there are always singular hypersurfaces in I_X . The proposition says that the singularities occur away from X .

0.5.11 Exercise. Verify that the proposition is true for the plane quartic, and to find the quadric singular at a point of the cubic curve.

A precise version of the complete intersection principle (0.5.6) would be that, e.g., in the case of [L4, 1.1], no hypersurfaces of degree d , singular at a general point $x \in X$, can osculate to order $2d + 1$ at x . If one understands “bending” in terms of genericity conditions on differentials invariants, then “bending enough” can be understood as the non-vanishing of certain contractions of differential invariants. For example, in the case $d = 2$, it is sufficient that the symmetrization map $T^* \otimes |\hat{II}| \rightarrow S^3 T^*$ is injective. (See §12.)

Taken together, the smoothness principle and the complete intersection principle indicate that perhaps varieties of small codimension must be complete intersections. Hartshorne has conjectured that if $a < \frac{n}{2}$ and X is smooth, then X must be a complete intersection. In fact, the two principles were developed in an attempt to understand Hartshorne’s conjecture and other work motivated by it from the perspective of projective differential geometry.

The following result is proved using the rank restriction theorem combined with a local study explained in §12:

0.5.11 Theorem [L6, 6.28]. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and let $x \in X$ be a general point. Let $b = \dim X_{sing}$. (Set $b = -1$ if X is smooth.) If $a < \frac{n-(b+1)+3}{3}$ then any quadric osculating to order four at x is smooth at x .*

By the discussion above, [L6, 6.28] implies

0.5.12 Corollary [L6, 6.29]. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety with I_X generated by quadrics. Let $b = \dim X_{sing}$. (Set $b = -1$ if X is smooth.) If $a < \frac{n-(b+1)+3}{3}$, then X is a complete intersection.*

0.6 Monge equations.

[L6, 6.29] above poses the question: *How can one determine if I_X is generated in degree two?* Before addressing the question of determining if X is contained in quadric hypersurfaces, let’s try an easier one: How can one tell if X is contained in a hyperplane H ?

$X \subset H$ if there exists a general point $x \in X$ such that $n_H \in \ker \mathbb{FF}_{X,x}^k$ for some k , where $n_H \in N_x^* X$ is a vector corresponding to H .

A corollary of the rank restriction theorem is that if $a < \frac{n-(b+1)}{2} + 1$, and $x \in X$ is a general point, then $III_{X,x} = 0$. Thus:

0.6.1 Theorem [L1]. *Let $X^n \subset \mathbb{CP}^{n+a}$ be a variety with $a < \frac{n-(b+1)}{2} + 1$ (where $b = \dim X_{\text{sing}}$). Let $x \in X$ be a general point. If a hyperplane H osculates to order two at x , then $X \subset H$.*

Thus in the situation of (0.6.1), to determine if X is contained in a hyperplane, two derivatives are sufficient.

Now consider the simplest case of a quadric hypersurface: Let $X \subset \mathbb{P}^2$ be a curve. How many derivatives does one need to take to determine whether or not X is a conic?

To fix a plane conic, one needs five points, or equivalently, one point and four derivatives. Thus to determine whether or not a given curve is a conic, it is necessary to take five derivatives (One point and four derivatives determines an osculating conic $C_x(X)$, the fifth derivative determines if $X = C_x(X)$). Plane conics given as a graph $y = f(x)$ are characterized by the classical *Monge equation*: $((y'')^{-\frac{2}{3}})''' = 0$. See [L4] for a derivation.

A generalization of the classical Monge equation to determine if an arbitrary variety is the intersection of quadrics would be impossible, because no fixed number of derivatives would suffice for all situations. However, if X is smooth and a is small, one could hope to have a fixed system. For example, if X is a smooth hypersurface and $n > 1$, Fubini [Fub] showed there exists a third order system characterizing quadric hypersurfaces, so the situation here is better than for curves.

It turns out that if X is of small codimension, but not a hypersurface, then one needs five derivatives. In §12 I derive a fifth order system of pde that I call the *generalized Monge system* that characterizes intersections of quadrics when $a < \frac{1}{3}(n - (b + 1) + 3)$.

0.7 Rigidity.

A general question related to step 5, of which (0.6) above is a special case, is to know how many derivatives one needs to take to recognize a given variety (or type of variety). In §13 I discuss several recognition questions centered around the results of [L9].

In [L9], I sharpen the result in step five of my proof of Zak's theorems on Severi varieties. I show that if X is a variety and $x \in X$ a general point such that $II_{X,x}$ is isomorphic to the second fundamental form at a point of a Severi variety other than $v_2(\mathbb{P}^2)$, then X is the corresponding Severi variety. (The result is false when $n = 2$, one must take third derivatives into account as well.) The case $n = 4$ had been conjectured by Griffiths and Harris in [GH]. I also prove the following theorems:

0.7.1 Theorem [L9]. *Let $X^{n+m} \subset \mathbb{P}^{nm+n+m-1+z}$, $n, m \geq 2$, be an open subset of a variety not contained in a hyperplane with the second fundamental form of the Segre $\mathbb{P}^n \times \mathbb{P}^m$ at general points. Then $z = 0$ and X is an open subset of the Segre $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^{nm+n+m-1}$.*

Note that the result is false if $n = m = 1$.

0.7.2 Theorem [L9]. *Let $X^{2(m-2)} \subset \mathbb{P}^{\binom{m}{2}-1+z}$, $m \geq 6$, be an open subset of a variety not contained in a hyperplane with the second fundamental form of the Grassmannian $G(2, m) \subset \mathbb{P}^{\binom{m}{2}-1}$ at general points. Then $z = 0$ and X is an open subset of the Grassmannian.*

Note that the result is false if $m < 5$.

The varieties above are examples of *Hermitian symmetric spaces*. A class of homogeneous varieties that resemble the Hermitian symmetric spaces are the *adjoint varieties*. For example, the variety of rank one and traceless $n \times n$ matrices is an adjoint variety. One might hope that the adjoint varieties are also determined by their second fundamental forms. This turns out not to be the case:

0.7.3 Theorem [LM2]. *There exist varieties having the same second fundamental form as the adjoint varieties at general points that are not adjoint varieties.*

0.8 Gauss maps.

Consider again the cylinder:

The tangent directions to its rulings are in $\text{singloc}[II]$. In fact, the embedded tangent space is not just constant to second order, but is constant all along the ruling, i.e., the ruling is a fiber of the Gauss map.

Another way to state the result (0.3.3.3) above is that if the Gauss map of X is degenerate, then X is singular. A proof of this statement and discussion of varieties with degenerate Gauss mappings is given in §5.

Let $Y \subset \mathbb{P}V$ be a smooth variety and let $X = \tau(Y)$ be the union of all embedded tangent lines to Y , the *tangential variety* of Y . Then X has a degenerate Gauss map because its tangent space is constant along the tangent lines of Y .

In [GH], Griffiths and Harris state that all varieties with degenerate Gauss maps are constructed from cones and (generalized) tangential varieties. They prove this statement when $\dim X = 2$. However when $\dim X = 3$, there are already counter-examples to their announcement. See §5 for some such examples.

0.9 Dual varieties.

When X is a hypersurface, $\gamma(X) \subset \mathbb{P}V^*$ is the set of hyperplanes tangent to X . A generalization of this situation is as follows:

Let $X^n \subset \mathbb{P}^{n+a} = \mathbb{P}V$. Define the *dual variety* $X^* \subset \mathbb{P}V^*$ as the set of hyperplanes tangent to X :

$$(0.9.1) \quad X^* = \overline{\{H \in \mathbb{P}V^* \mid \exists x \in X_{sm} \text{ such that } \tilde{T}_x X \subseteq H\}}.$$

One expects X^* to be a hypersurface, because there is an $(a-1)$ -dimensional space of hyperplanes tangent to each point and an n dimensions of points. Let $\delta_* = \delta_*(X) = n + a - 1 - \dim X^*$; the *dual defect* of X . Zak proved that if X is smooth and not contained in a hyperplane, then $\delta_* \leq a - 1$ (see §7).

A standard fact is that $(X^*)^* = X$, so we may think of X^* as a transform of X . In fact, the dual variety is the generalization to algebraic geometry of the Legendre transform in classical mechanics. Transforms are useful because they reorganize data in such a way that information one looks for in an object becomes more easily accessible in its transform. For example, the Fourier transform exchanges global and local data. In my work with B. Ilic, [IL], our perspective was to view X^* as a transform of X . We describe ways in which the global geometry of X is reflected in the local geometry of X^* . We prove

an inversion formula that shows that the second fundamental form at a point $H \in X^*$ contains information about all the points to which H is tangent.

Using the inversion formula, we show:

0.9.2 Theorem [IL]. *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety with dual defect δ_* . If $H \in X_{sm}^*$, then $|II_{X^*,H}|$ is a system of quadrics of projective dimension δ_* and constant rank $n - \delta_*$.*

Compare this result to an earlier result in [GH] (Bertini??) that if X is any variety and $x \in X$ is a general point, then $|II_{X,x}|$ is a system of bounded rank $n - \delta_*$.

(0.9.2) led us to examine systems of quadrics of constant rank. We were able to solve an old question: What is the maximum dimension of a system of quadrics of constant rank r on \mathbb{C}^m ? (The answer when r is odd was known classically to be one, see §8). We showed:

0.9.3 Theorem [IL, 2.16]. *If r is even, then*

$$\max \{ \dim(A) \mid A \subset S^2 \mathbb{C}^m \text{ is of constant rank } r \} = m - r + 1.$$

[IL, 2.16] and [IL, 3.24] together furnish a new proof of Zak's theorem that $\delta_* \leq a - 1$.

0.10 Ruled and uniruled varieties.

A variety $X \subset \mathbb{P}V$ is *uniruled* by k -planes if through each $x \in X$ there exists a k -plane contained in X passing through x . X is *ruled* if it can be described as a fibration over a base space that parametrizes the k planes, e.g., X is ruled if there is a unique k -plane through each point. The quadric hypersurface for $n > 3$ is uniruled but not ruled.

Consider again the hyperbola, a ruled surface.

(0.10.1)

The tangent directions to the ruling are in $\text{Base}|II|$. Directions in $\text{Base}|II|$ of an arbitrary variety X are not usually tangent to linear spaces contained in X . In [L7] I determine additional conditions that imply a variety is uniruled by k -planes. In coordinates, the question is: How many derivatives are needed to determine if X is uniruled? For example, given a surface in \mathbb{P}^3 , there are always at least two tangent directions in $\text{Base}|II|$ (so in this aspect, the hyperbola is not special at all). Such directions are called *asymptotic directions* in classical differential geometry. So two derivatives are not enough to see if a surface is ruled. A classical result [Blaschke ??] states that three derivatives are enough. Here is a generalization of the classical result:

0.10.2 Theorem [L7]. *Let $X^n \subset \mathbb{A}^{n+a}$ or $X^n \subset \mathbb{P}^{n+a}$ be an open subset of a variety of a smooth (respectively analytic) submanifold of an affine or projective space such that at every point (resp. at a general point) there is a line osculating to order $n + 1$. Then X is uniruled by lines.*

There exist analytic open subsets of varieties $X^n \subset \mathbb{A}^{n+1}$ or $X^n \subset \mathbb{P}^{n+1}$ having a line osculating to order n at every point that are not uniruled. Over \mathbb{C} , every hypersurface has this property.

(0.10.2) also sharpens Z. Ran's dimension+2 secant lemma [R2]. Ran proves that if a variety has lines osculating to order $n + 1$ at each point, then the union of the osculating

lines is at most $n + 1$ dimensional. The above result shows that the union is in fact n dimensional.

In §11 I present some preliminary results regarding the integer $m = m(n, a, k)$ such that any open subset of a variety $X^n \subset \mathbb{P}^{n+a}$ having k -planes osculating to order m at each point must be uniruled by k planes, but that there exist open subsets of varieties with k planes osculating to order $m - 1$ that are not uniruled.

In a similar vein, certain geometric situations when osculation to order two is sufficient to imply containment of a linear space are presented in §7.

0.11 Aside.

I spend a lot of time studying systems of quadrics, so people occasionally ask me questions about them. An amusing question was posed to me by M. Kontsevich: Let $A = (a_j^i)$ be an orthogonal matrix with no entries zero, form the *Hadamard inverse* of A , $Hinv(A)$, by $(Hinv(A))_j^i = 1/a_j^i$. He conjectured that the rank of $Hinv(A)$ was never two (but that perhaps all other ranks were possible).

A classical theorem regarding systems of quadrics is Castelnuovo's lemma, which states that if $2n + 3$ points lie on an $\binom{n}{2}$ -dimensional system of quadrics, then the points must lie on a rational normal curve.

I resolve Kontsevich's conjecture by translating it to a problem in algebraic geometry that turned out to be a variant of Castelnuovo's lemma: If a self-associated point set of $2n + 2$ points in \mathbb{P}^n (which necessarily lie on a $\binom{n}{2}$ -dimensional space of quadrics) is such that the standard Cremona transform of $n + 1$ of the points with respect to the simplex formed by the other $n + 1$ points is contained in a \mathbb{P}^2 , then in fact the $2n + 2$ points are collinear. (see [L8] for details).

0.12 Self-similarity of geometric issues under change of scale.

When working with infinitesimal invariants, similar issues often arise as when working globally. To give three examples:

Ilic and I used the Lefschetz theorem on hyperplane sections to prove the constant rank theorem, which we used to obtain an infinitesimal proof of Zak's bound on the dual defect. Zak's original proof rests on the Fulton-Hansen connectedness theorem, which in turn relies on an extension of the Lefschetz theorem due to Goresky and MacPherson (see §3).

A second example is that if X is not a complete intersection, I_X must have syzygies. (For example the quadrics defining the Segre have linear syzygies that arise from expanding out the three by three minors.) If X is not a complete intersection, its differential invariants also must have syzygies. I hope that these syzygies will be easier to study because their existence can be related to degeneracies of auxiliary varieties.

A third example is the following proposition in [LM1], where secant varieties arise in the study of infinitesimal geometry:

0.12.1 Proposition [LM1]. *Let $X^n \subset \mathbb{P}^{n+a}$ be an open subset of a variety and let $x \in X$ be a general point. Then $\text{Base}|\mathbb{FF}^k| \supseteq \sigma_k(\text{Base}|II|)$.*

$\sigma_k(X)$ denotes the closure of the union of all secant $(k - 1)$ -planes to X .

0.13 Notation.

I will use the following conventions for indices

$$\begin{aligned} 0 &\leq B, C \leq n+a \\ 1 &\leq \alpha, \beta \leq n \\ n+1 &\leq \mu, \nu \leq n+a \end{aligned}$$

Alternating products of vectors will be denoted with a wedge (\wedge), and symmetric products will not have any symbol (e.g. $\omega \circ \beta$ will be denoted $\omega\beta$). $T_x X$ denotes the holomorphic tangent space to X at x and $\tilde{T}_x X$ the embedded tangent space. I often suppress reference to X and x , abbreviating the names of bundles, e.g. T should be read as $T_x X$, N as $N_x X$ etc... If $v \in \hat{T}_{[w]} X$, then I write $\underline{v} := v \otimes w^*$ mod $w \in T_{[w]} X$, where v, w a part of a basis of V and w^* denotes the dual basis vector to w . $\{e_i\}$ means the span of the vectors e_i over the index range i . If $Y \subset \mathbb{P}^m$ then $\hat{Y} \subset \mathbb{C}^{m+1}$ will be used to denote the cone over Y (with the exception that the cone over the embedded tangent space \tilde{T} will be denoted \hat{T}). If $A \in \mathbb{C}^{n+a+1}$, its projection to \mathbb{P}^{n+a} will be denoted $[A]$. If V is a vector space and W a subspace, and (e_1, \dots, e_n) a basis of V such that $\{e_1, \dots, e_p\} = W$, I write $\{e_{p+1}, \dots, e_n\}$ mod W to denote the space V/W . For vector subspaces $W \subset V$, I use the notation $W^\perp \subset V^*$ for the annihilator of W in V^* . I will use the summation convention throughout (i.e. repeated indices are to be summed over). $\mathfrak{S}_{\alpha\beta\gamma}$ denotes cyclic summation over the fixed indices $\alpha\beta\gamma$. In general, X will denote a variety, X_{sm} its smooth points, and X_{sing} its singular points. \mathbb{CP}^k will be denoted \mathbb{P}^k . \mathbb{FF}_X^k is the k -th fundamental form of X . I often denote \mathbb{FF}_X^2 by II and \mathbb{FF}_X^3 by III . $F_k = F_k^\mu e_\mu$ mod \hat{T} is the differential invariant called the $(k-2)$ -nd variation of II . By a *general point* $x \in X$ I mean a smooth point of X such that all the discrete information in the differential invariants of X is locally constant. The nongeneral points of X are a codimension one subset of X .

§1. EXAMPLES OF HOMOGENEOUS VARIETIES AND THEIR USES

A smooth projective variety $X \subset \mathbb{P}V$ is *homogeneous* if X is the closed orbit of a (complex semi-simple) Lie group G acting on $\mathbb{P}V$. Such X can be described intrinsically as $X = G/P$ where P is a subgroup of G , called a *parabolic* subgroup.

One can reduce to the case of varieties that are orbits of simple groups (those whose Lie algebras have no nontrivial ideals), as others are just products of such. There are three simple groups occurring in series, $SL(V, \Omega)$, $Sp(V, \omega)$, $O(V, Q)$, where respectively $\Omega \in \Lambda^m V^*$, $\omega \in \Lambda^2 V^*$, $Q \in S^2 V^*$ are nondegenerate elements and the groups are the subgroups of $GL(V)$ preserving the forms ($\dim V = m$). Actually $O(V, Q)$ is not connected or simply connected, its connected component of the identity is called $SO(V, Q)$, its simply connected double cover is called $Spin(V, Q)$. For $Sp(V, \omega)$, the dimension of V must be even to have a nondegenerate two-form. Since the behaviour of nondegenerate quadratic forms is quite different in even and odd dimensions (e.g. every rotation in \mathbb{R}^{2n+1} has a fixed axis), for $O(V, Q)$, the case where $\dim V$ is even and odd are considered as different groups.

In addition to these groups, there are five *exceptional* groups, which are called G_2 , F_4 , E_6 , E_7 , E_8 . In what follows I will describe all but the last two. (Actually G_2 is quite easy to describe, it is the subgroup of $GL(7, \mathbb{C})$ preserving a nondegenerate (generic) element of $\Lambda^3 \mathbb{C}^7$.)

1.1 More on Segre varieties.

Given vector spaces W_1, \dots, W_r , define an embedding $X = \text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_r) \subset \mathbb{P}(W_1 \otimes \dots \otimes W_r)$ by $[w_1] \times \dots \times [w_r] \mapsto [w_1 \otimes \dots \otimes w_r]$. X is called the *Segre* embedding of $\mathbb{P}W_1 \times \dots \times \mathbb{P}W_r$. (In the language of [GKZ], these are the “rank one multidimensional matrices”.)

One can form the product of varieties. Unlike the affine case where the product of $M \subset \mathbb{A}^m$ and $N \subset \mathbb{A}^n$ is naturally $M \times N \subset \mathbb{A}^{m+n}$, the product to two projective varieties naturally occurs as a subvariety of the Segre of their ambient spaces. If $X \subset \mathbb{P}V$, $Y \subset \mathbb{P}W$, $\text{Seg}(X \times Y) \subset \mathbb{P}(V \otimes W)$ is the natural projective embedding of $X \times Y$.

If $W_1 = W_2$, one can consider instead of arbitrary matrices, symmetric or skew-symmetric matrices of minimal rank and their generalizations. We do so in the following two examples.

1.2 Grassmannians.

Let V be a vector space, choose an identification $V \simeq \mathbb{C}^m$ so $\Lambda^2 V$ is identified with the $m \times m$ skew symmetric matrices. Let $G(2, V) \subset \mathbb{P}(\Lambda^2 V)$ denote the projectivization of the rank two matrices (i.e. the matrices of minimal rank, as the rank of a skew-symmetric matrix is always even). $G(2, V)$ is called the *Grassmannian of two planes in V* .

Let e_1, \dots, e_n be a basis of V , then $\{e_i \wedge e_j \mid i \leq j\}$, is a basis of $\Lambda^2 V$, and we may think of $e_i \wedge e_j$ as the skew symmetric matrix with 1 in the (i, j) -th slot, -1 in the (j, i) -th slot and zero elsewhere. Any $E \in G(2, V)$ can be written $E = v \wedge w = v^i w^j (e_i \wedge e_j)$, where $v = v^i e_i, w = w^j e_j$. This gives two interpretations of $G(2, V)$; as the decomposable elements in $\Lambda^2 V$ (i.e. elements of the form $v \wedge w$, with $v, w \in V$), and as the set of two planes through the origin in V (E is the two-plane spanned by v and w).

Generalizing, let $G(k, V) \subset \mathbb{P}\Lambda^k V$ denote the set of k -planes through the origin in V , or equivalently the decomposable k -vectors (those that can be written $v^1 \wedge \dots \wedge v^k$, with each $v^j \in V$). We may also think of $G(k, V)$ as the space of \mathbb{P}^{k-1} 's in $\mathbb{P}V$. When we use this perspective, we will write $G(k, V) = \mathbb{G}(k-1, \mathbb{P}V)$.

The tangent space to any manifold at a point is a vector space, the tangent space to $G(k, V)$ is a vector space with additional structure, namely $T_E G(k, V) \simeq \text{Hom}(E, V/E) = E^* \otimes (V/E)$. To see this take a curve $E(t) = v^1(t) \wedge \dots \wedge v^k(t)$ and differentiate at $t = 0$. The tangent space to any homogeneous space is always a vector space with additional structure, and this additional structure can be deduced either intrinsically or extrinsically.

In particular for the case of $G(1, V) = \mathbb{P}V$, $T_x \mathbb{P}V = \hat{x}^* \otimes (V/\hat{x})$. (We use the notation that for $Y \subset \mathbb{P}V$, $\hat{Y} \subset V$ is the corresponding cone.) If $X \subset \mathbb{P}V$ is a subvariety, the (intrinsic holomorphic) tangent space to X inherits this additional structure. Letting $\tilde{T}_x X$ denote the embedded tangent space and $\hat{T}_x X = \hat{\tilde{T}}_x X$, then $T_x X = \hat{x}^* \otimes (\hat{T}_x X/\hat{x})$. Since we are not using a metric, the normal bundle is just a quotient bundle, $N_x X = T_x \mathbb{P}V / T_x X = \hat{x}^* \otimes (V/\hat{T}_x X)$. Traditionally the line bundle with fiber $\hat{x}^{\otimes k}$ is denoted $\mathcal{O}_{\mathbb{P}V}(-k)$ and if E is any vector bundle, $E(k) = E \otimes \mathcal{O}(k)$.

Note that $G(k, V) = G(n-k, V^*)$ as specifying a k plane $E \subset V$ is equivalent to specifying its annihilator $E^\perp \subset V^*$. Of particular importance is $G(n-1, V) = \mathbb{P}V^*$ the *dual projective space*, where points of $\mathbb{P}V^*$ correspond to hyperplanes in $\mathbb{P}V$.

If $G \subset SL(V)$ is a group preserving additional structure (e.g. a quadratic form Q or symplectic form ω) one can define the corresponding null Grassmannians, e.g.

$$(1.2.1) \quad \begin{aligned} G_{Q\text{-}null}(k, V) &:= \{E \in G(k, V) \mid Q(v, w) = 0 \forall v, w \in E\} \\ G_{\omega\text{-}null}(k, V) &:= \{E \in G(k, V) \mid \omega(v, w) = 0 \forall v, w \in E\} \end{aligned}$$

The Q -null Grassmannians are naturally embedded in $\mathbb{P}\Lambda^k V$. In the case $\dim V = 2m$ and $k = m$, $G_{Q\text{-}null}(m, 2m)$ has two isomorphic components. The components are called the *Spinor varieties* \mathbb{S}_m and each \mathbb{S}_m embeds into a smaller projective space which I will describe after explaining Clifford algebras.

1.2.2 Exercise. Show that $G_{\omega\text{-}null}(k, V)$ naturally embeds into a linear subspace of $\mathbb{P}(\Lambda^k V)$. Calculate its tangent space at a point.

1.3 Veroneses.

Let $S^2 V$ denote the symmetric matrices and let $v_2(\mathbb{P}V) \subset \mathbb{P}(S^2 V)$ denote the projectivization of the rank one elements. $v_2(\mathbb{P}V)$ is the image of $\mathbb{P}V$ under the injective mapping

$$(1.3.1) \quad \begin{aligned} v_2 : \mathbb{P}V &\rightarrow \mathbb{P}S^2 V \\ [x] &\mapsto [x \circ x] \end{aligned}$$

The d -th Veronese embedding of $\mathbb{P}V$, $v_d(\mathbb{P}V) \subset \mathbb{P}S^d V$ is defined by $v_d(x) = x^d = x \circ \dots \circ x$. Given $X \subset \mathbb{P}V$, we can consider the Veronese re-embeddings of X , $v_d(X) \subset \mathbb{P}(S^d V)$, which will turn out to be useful in our study of complete intersections. Note that if Z is a hypersurface of degree d , $v_d(Z) = H_Z \cap v_d(\mathbb{P}V)$, where H_Z is the hyperplane associated to the equation of Z .

1.4 Division algebras and the spinor variety \mathbb{S}_5 .

There are four division algebras over \mathbb{R} : \mathbb{R} , \mathbb{C} , \mathbb{H} , and \mathbb{O} (where \mathbb{O} denotes the *octonians*, or *Cayley numbers*). The octonians are similar to the quaternions. If one thinks of $u \in \mathbb{H}$ as $u = u^0 + u^1\epsilon_1 + u^2\epsilon_2 + u^3\epsilon_3$ where $u^i \in \mathbb{R}$ and the ϵ_i satisfy $\epsilon_i^2 = -1$ and the following multiplication table:

$$(1.4.1)$$

(to be read...), then given $u \in \mathbb{O}$, write $u = u^0 + u^1\epsilon_1 + \dots + u^7\epsilon_7$ where $\epsilon_j^2 = -1$ and the ϵ_j satisfy the following multiplication table:

Let $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or \mathbb{O} and let $\mathbb{A} = \mathbb{A}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The only spinor variety we will have immediate need of is $\mathbb{S}_5^{10} \subset \mathbb{P}^{15}$ which may be described as follows: Let $\mathbb{C}^{16} = \mathbb{O}^2 \otimes_{\mathbb{R}} \mathbb{C}$ have octonionic coordinates u, v . \mathbb{S}_5 is defined by the equations $u\bar{u} = 0, v\bar{v} = 0, u\bar{v} = 0$ where the last is eight equations.

1.4.2 Exercise. what are the corresponding varieties for the other division algebras?)

Let $Aut(\mathbb{A}) := \{g \in GL(\mathbb{A}) \mid (gu)(gv) = g(uv) \forall u, v \in \mathbb{A}\}$. $Aut(\mathbb{A}) \subset GL(\text{Im}\mathbb{A})$ is respectively $\{Id\}, \mathbb{Z}_2, Sl_2\mathbb{C}, G_2$, providing a second definition of G_2 .

1.5 Clifford algebras.

Let (V, Q) be as above. Given any linear subspace $L \subset V$, one can define $L^{\perp Q} \subset V$, by $L^{\perp Q} = \{w \in V \mid Q(v, w) = 0 \forall v \in L\}$. If $Q|_L$ is nondegenerate $V = L \oplus L^{\perp Q}$. In this case, for all $v \in V$, we may write $v = v_1 + v_2$ with $v_1 \in L, v_2 \in L^{\perp Q}$. We may define the reflection of v in L by $refl_L(v) = v_1 - v_2$.

Recall that $O(V, Q)$ is the subgroup of $GL(V)$ preserving Q , and $SO(V, Q)$ is the component of $O(V, Q)$ containing the identity.

1.5.1 Theorem, Cartan-Diedonne (see [Hv]). $O(V, Q)$ is the group generated by reflections in lines. $SO(V, Q)$ is the group generated by even numbers of reflections. More precisely, $O(V, Q) = \{refl_{l_1} \circ \dots \circ refl_{l_k} \mid l_j \in V\}$ and we may assume $k \leq n$. Similarly for $SO(V, Q)$, only k must be even.

To define $Spin(V, Q)$, the connected and simply connected group corresponding to $O(V, Q)$, we will need to generalize the notion of a reflection. Let $\Lambda^\bullet V = \otimes V / \{x \otimes y - y \otimes x\}$ be the exterior algebra. The exterior product in $\Lambda^\bullet V$, $(x, y) \mapsto x \wedge y$ may be interpreted as follows: Let $\hat{G}(i, V) \subset \Lambda^i V$ denote the cone over the Grassmannian. If $x \in \hat{G}(i, V)$, $y \in \hat{G}(j, V)$, then $x \wedge y \in \hat{G}(i+j, V) \subset \Lambda^{i+j} V$ represents the $i+j$ -plane spanned by x and y . If $x \in V$, then $x \wedge y$ is analogous to the component of x in $[y]^{\perp Q}$. If y has unit length, then $\|x \wedge y\|_Q = \|proj_{y^{\perp Q}}(x)\|_Q$. Note that we do not need Q to define $x \wedge y$.

Let $x \lrcorner y$ be defined to be the Q -adjoint of $x \wedge y$, that is $Q(x \wedge y, z) = Q(y, x \wedge z)$ for all $x, y, z \in \Lambda^\bullet V$. If $x \in V$ and $y \in \hat{G}(j, V)$, then $x \lrcorner y$ is analogous to the component of x in $\mathbb{P}\{y\}$. Note that if y has unit length, then $\|x \lrcorner y\|_Q = \|proj_y(x)\|_Q$.

Consider, for $x \in V, y \in \Lambda^\bullet V$,

$$x \circ y := x \lrcorner y - x \wedge y$$

which can be thought of the “generalized reflection” of x in y . We can, by linearity, extend \circ to a multiplication on $\Lambda^\bullet V$.

1.5.2 Definition. Let V be a vector space with a nondegenerate quadratic form Q . Let $Clifford(V, Q) = Cl(V, Q) := (\Lambda^\bullet V, \circ)$, the *Clifford algebra* of (V, Q) .

As is usual with algebras formed from vector spaces, we have a fundamental classifying/universality lemma:

1.5.3 Fundamental Lemma of Clifford algebras (see [Hv]). Let V be a vector space with a nondegenerate quadratic form Q and let \mathcal{A} be an associative algebra with unit. If $\phi : V \rightarrow \mathcal{A}$ is a mapping such that for all $x, y \in V$

$$\phi(x)\phi(y) + \phi(y)\phi(x) = 2Q(x, y)Id_{\mathcal{A}}$$

then ϕ has a unique extension to an algebra mapping $\tilde{\phi} : Cl(V, Q) \rightarrow \mathcal{A}$.

1.5.4 Exercise. Verify that $\mathcal{A} = (\Lambda^\bullet V, \circ)$ satisfies the hypotheses of the fundamental lemma.

In $Cl(V, Q) = (\Lambda^\bullet V, \circ)$, the degree of a form is no longer well defined, but there is still a notion of parity. Let $Cl^{even}(V, Q), Cl^{odd}(V, Q) \subset Cl(V, Q)$ denote the corresponding even and odd subspaces. As vector spaces, $Cl^{even}(V, Q) = \Lambda^{even}V$, $Cl^{odd}(V, Q) = \Lambda^{odd}V$.

Let $Cl^*(V, Q) \subset Clifford(V, Q)$ denote the invertible elements.

1.5.5 Definition.

$$Pin(V, Q) := \{a \in Cl^*(V, Q) \mid a = u_1 \circ \dots \circ u_r, u_j \in V, Q(u_j, u_j) = 1\}$$

$$Spin(V, Q) := \{a \in Pin(V, Q) \mid r \text{ is even}\}.$$

From now on, assume $\dim V = 2m$.

We have a description of $Spin(V, Q)$ as an abstract group and as a group acting on a vector space, namely $\Lambda^\bullet V$. This action is clearly reducible because it preserves parity, but in fact it reduces further. Fix $U^m \subset V$, a maximal null subspace. It is easy to verify that in fact $Spin(V, Q)$ preserves $\Lambda^{even}U \subset \Lambda^\bullet V$ and, slightly more difficult to verify, but also true, that this action is irreducible. (Traditionally, $\Lambda^{even}U$ is denoted \mathcal{S}^+ and called the space of *positive spinors*.)

Given $a = u_1 \circ \dots \circ u_r \in Clifford(V, Q)$, let $\tilde{a} = (-1)^r u_r \circ \dots \circ u_1$. This is a well defined involution. Using it, we may define an action ρ of $Spin(V, Q)$ on V , by $\rho(a)v := av\tilde{a}$, and it is easy to verify that ρ is a $2 : 1$ map $Spin(V, Q) \rightarrow SO(V, Q)$.

One may describe $\mathbb{S}_m \subset \mathbb{P}\mathcal{S}^+$, as the null m -planes E such that $\dim(E \cap U) - n \equiv 1 \pmod{2}$ as follows:

Define a map

$$(1.5.6) \quad \begin{aligned} V \times \Lambda^{even}U &\rightarrow \Lambda^{odd}U \\ (v, \alpha) &\mapsto \text{proj}_{\Lambda^{odd}U}(v \circ \alpha) \end{aligned}$$

($v \circ \alpha \in \Lambda^{odd}V$ and then we project it to $\Lambda^{odd}U$). In other words, for $\alpha \in \Lambda^{even}U$, we have a map $L_\alpha : V \rightarrow \Lambda^{odd}U$. In particular, for $1 \in \Lambda^0U$, $\ker L_1 = U'$, where $U' \subset V$ is the unique null m -plane such that $V = U \oplus U'$. Since the map is $Spin(V, Q)$ equivariant, for $g \in Spin(V, Q)$, $\ker L_g = \ker L_{g \circ 1} = \rho(g)(\ker L_1) = gU'\tilde{g}$. In summary:

1.5.7 Proposition/Definition.

$$\begin{aligned} \mathbb{S}_m(V, Q, [U]) &= \{\mathbb{P}^{m-1} \subset Q \mid \dim(\mathbb{P}^{m-1} \cap \mathbb{P}U) - m \equiv 1 \pmod{2}\} \\ &= \mathbb{P}\{\text{the } Spin(V, Q) \text{ orbit of } 1 \in \Lambda^{even}U\} \subset \mathbb{P}(\Lambda^{even}U). \end{aligned}$$

In §4 I show that $T_{[1]}\mathbb{S}_m \simeq \Lambda^2U$; in particular, $\dim \mathbb{S}_m = \binom{n}{2}$.

1.6 The Severi varieties.

Here is a different generalization of the Veronese $v_2(\mathbb{P}V)$: Let \mathbb{A} denote the complexification of a division algebra as above, and let $\mathcal{H}_{\mathbb{R}}$ denote the $\mathbb{A}_{\mathbb{R}}$ -Hermitian forms on $\mathbb{A}_{\mathbb{R}}^3$, i.e., the $3 \times 3 \mathbb{A}_{\mathbb{R}}$ -Hermitian matrices. If $x \in \mathcal{H}_{\mathbb{R}}$, then we may write

$$(1.6.1) \quad x = \begin{pmatrix} r_1 & \bar{u}_1 & \bar{u}_2 \\ u_1 & r_2 & \bar{u}_3 \\ u_2 & u_3 & r_3 \end{pmatrix} \quad r_i \in \mathbb{R}, \quad u_i \in \mathbb{A}_{\mathbb{R}}.$$

Let $\mathcal{H} = \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

1.6.2 Exercise. Verify that the notion of x^2 and x^3 make sense (one needs to use the Moufang identites in the case of the octonians, see [Hv]).

Define a cubic form \det on \mathcal{H} by

$$(1.6.3) \quad \det(x) := \frac{1}{6}((\text{trace}(x))^3 + 2\text{trace}(x^3) - 3\text{trace}(x)\text{trace}(x^2)).$$

\det is just the usual determinant of a 3×3 matrix when $\mathbb{A} = \mathbb{C}$. When $\mathbb{A}_{\mathbb{R}} = \mathbb{O}$ one cannot define \det for 4×4 or larger matrices.

Now, considering \mathcal{H} as a vector space over \mathbb{C} , let G be the subgroup of $GL(\mathcal{H}, \mathbb{C})$ preserving \det , i.e., define

$$G := \{g \in GL(\mathcal{H}, \mathbb{C}) \mid \det(gx) = \det(x) \ \forall x \in \mathcal{H}\}.$$

The respective groups are:

$$(1.6.4) \quad \begin{array}{ll} \mathbb{A}_{\mathbb{R}} = & G = \\ \mathbb{R} & Sl(3, \mathbb{R})^{\mathbb{C}} = Sl(3, \mathbb{C}) \\ \mathbb{C} & Sl(3, \mathbb{C})^{\mathbb{C}} = Sl(3, \mathbb{C}) \times Sl(3, \mathbb{C}) \\ \mathbb{H} & Sl(3, \mathbb{H})^{\mathbb{C}} = Sl(6, \mathbb{C}) \\ \mathbb{O} & 'Sl(3, \mathbb{O})^{\mathbb{C}}, = E_6 \end{array}$$

where I write ' $Sl(3, \mathbb{O})^{\mathbb{C}}$ ' merely to be suggestive. We take the above as the definition of E_6 . The group F_4 is defined to be the subgroup of E_6 preserving the quadratic form $Q(x, x) = \text{trace}(x^2)$ where xy is the usual matrix multiplication and one must again use the Moufang identities to be sure Q is well defined.

1.6.5 Exercise. Show that the action of F_4 on \mathcal{H} preserves the line $\mathbb{C}\{Id\}$ and $\mathcal{H}_0 := \{x \in \mathcal{H} \mid \text{trace}(x) = 0\}$. (In fact, F_4 acts irreducibly on both factors.)

\det tells us which elements of \mathcal{H} are of less than full rank. One can also unambiguously define a notion of being rank one; either by taking 2×2 minors or by noting that under the G action each $x \in \mathcal{H}$ is diagonalizable and one can take as the rank of x the number of nonzero elements in the diagonalization of x .

Let

$$X := \mathbb{P}\{ \text{rank one elements of } \mathcal{H} \} = \mathbb{P}\{G \text{ orbit of any rank one matrix}\}.$$

$X = (\mathbb{A}_{\mathbb{R}}\mathbb{P}^2)^{\mathbb{C}}$, that is, the complexification of the space of $\mathbb{A}_{\mathbb{R}}$ -lines in $\mathbb{A}_{\mathbb{R}}^3$. The four varieties $X \subset \mathbb{P}\mathcal{H}$ are called the *Severi varieties*. We have already seen the first three, they are $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$, $Seg(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$, and $G(2, 6) \subset \mathbb{P}^{14}$.

1.6.6 Exercise. What are the analogous groups and varieties if one takes instead the 2×2 \mathbb{A} -Hermitian forms?

1.6.7 Exercise. Let $\mathbb{O}\mathbb{P}_0^2 = \mathcal{H}_0 \cap \mathbb{O}\mathbb{P}^2$. Show that

$$\mathbb{O}\mathbb{P}_0^2 = \mathbb{P}\{x \in \mathcal{H}_0 \mid x^2 = 0\}$$

and deduce that it is a homogeneous space of F_4 .

§2. CONSTRUCTING NEW VARIETIES FROM OLD

As mentioned in the introduction, one way to study subtle properties of a variety $X^n \subset \mathbb{P}V$ is to study coarse properties of auxilliary varieties one constructs from X . We will be primarily concerned with studying the dimensions of auxilliary varieties. Here are some constructions:

2.1 The Gauss images of X .

We continue the notations $V = \mathbb{C}^{n+a+1}$ and $X^n \subset \mathbb{P}V$ a subvariety.

The *Gauss map* of $X \subset \mathbb{P}V$ is defined at smooth points by:

$$\begin{aligned}\gamma : X_{sm} &\rightarrow \mathbb{G}(n, \mathbb{P}V) \\ x &\mapsto \tilde{T}_x X.\end{aligned}$$

γ can be completed to a rational map $X \dashrightarrow \mathbb{G}(n, \mathbb{P}V)$ whose image we denote $\gamma(X)$, the *Gauss image* of X . We will say γ is *degenerate* if $\dim \gamma(X) < \dim X$.

More generally, for each $x \in X_{sm}$ we can consider the $(n+k)$ -dimensional projective spaces that contain $\tilde{T}_x X$, and the resulting submanifold of $\mathbb{G}(n+k, \mathbb{P}V)$. The closure of these is a variety which we denote $\gamma_k(X)$ and call the k -th *Gauss image* of X . The notation is such that $\gamma_0(X) = \gamma(X)$. Of particular interest is the *dual variety* $X^* = \gamma_{a-1}(X) \subset \mathbb{G}(n+a-1, \mathbb{P}V) = \mathbb{P}V^*$. One expects $\dim \gamma_k(X)$ to be $n+k(a-k)$ because there are n dimensions of points on X and a $k(a-k)$ -dimensional space of $(n+k)$ -planes tangent to each smooth point. We say $\gamma_k(X)$ is *degenerate* if it fails to be of the expected dimension. We will be particularly interested in the degeneracy of the dual variety, we let $\delta_* = n+a-1-\dim X^*$ denote the *dual defect* of X .

To better understand the higher Gauss images, it is useful to use a standard construction in algebraic geometry, the *incidence correspondence*.

Let

$$(2.1.1) \quad \mathcal{I} = \mathcal{I}(X, \mathbb{G}(n+k, \mathbb{P}V)) = \overline{\{(x, L) \mid \tilde{T}_x X \subseteq L\}}.$$

We have the following picture

$$(2.1.2) \quad \mathcal{I}_k$$

$$X \qquad \mathbb{G}(n+k, \mathbb{P}V)$$

maps are π_k, ρ_k *

By definition $\gamma_k(X) = \rho_k \pi_k^{-1}(X)$. We will often use the notation $Y_L = \pi(\rho^{-1}L)$. In the most important case of \mathcal{I}_{a-1} we write $\mathcal{I} = \mathcal{I}_{a-1}$ etc...

With an eye towards Morse theory (which is used in the proof of the Lefschetz theorems, see §3), one can define X^* as follows: Consider the pairing $\langle \cdot, \cdot \rangle : \hat{X} \times V^* \rightarrow \mathbb{C}$. For $p \in \hat{X}$, consider the function

$$(2.1.3) \quad \begin{aligned}\langle p, \cdot \rangle : V^* &\rightarrow \mathbb{C} \\ q &\mapsto \langle p, q \rangle.\end{aligned}$$

Then $\pi^{-1}(x) = \mathbb{P}(\text{critical point of } \langle p_x, \cdot \rangle)$, where $p_x \in \hat{x}$ and

$$(2.1.4) \quad X^* = \overline{\cup_{x \in X_{sm}} \mathbb{P}(\text{critical points of } \langle p_x, \cdot \rangle)}$$

$\tilde{T}_x X \subseteq H$ if and only if $X \cap H$ is singular at x , so another definition of X^* is the union of the hyperplanes H such that $X \cap H$ is singular.

2.2 Secant varieties and joins.

Let $Y, Z \subset \mathbb{P}V$ be two varieties. We define the *join* of Y and Z ,

$$S(Y, Z) = \overline{\cup_{y \in Y, z \in Z} \mathbb{P}_{yz}^1}$$

where the closure is not necessary if the two varieties do not intersect. This generalizes the secant variety which is the case $Y = Z$. We can similarly form the join of k varieties Y_1, \dots, Y_k ,

$$(2.2.1) \quad S(Y_1, \dots, Y_k) = \overline{\cup_{y_j \in Y_j} \mathbb{P}_{y_1, \dots, y_k}^{k-1}}$$

A special case of this construction is if $Z = L$ is a linear space. Then $S(Y, L)$ is a cone over Y .

An important result about joins is the following:

2.2.2 Terracini Lemma. *Let $Y, Z \subset \mathbb{P}V$ be varieties and let $x \in \mathbb{P}_{yz}^1$. Then*

$$\hat{T}_x S(Y, Z) \supseteq \hat{T}_y Y + \hat{T}_z Z.$$

Moreover, if x is a general point of $S(Y, Z)$, then equality holds.

2.2.3 Exercise. Prove the inclusion part of Terracini's lemma. Hint: consider the two curves $p + q(t)$ and $p(t) + q$, where $p \in Y$, $q \in Z$ and differentiate.

2.2.4 Special cases.

If $Z = L$ is a linear space, then $\tilde{T}_x S(Y, L)$ contains L for all $x \in S(Y, L)_{sm}$ and $S(Y, L)$ is the cone over L .

If $Y_j = Y$ for all j , we call $S(Y, \dots, Y)$ the *k -th secant variety of Y* and use the notation $\sigma_k(Y)$. The notation is such that $\sigma_1(Y) = Y$. We often denote $\sigma_2(Y)$ by $\sigma(Y)$. Note that one expects $\dim \sigma_k(Y) = \min \{nk + k - 1, n + a\}$ as there are nk dimensions worth of picking k points on Y and their span is a \mathbb{P}^{k-1} .

A related notion is the *k -th multi-secant variety of X* , $MS_k(X)$ is defined to be the closure of the union of all lines in $\mathbb{P}V$ containing k points of X . An essential observation regarding them, due to Severi, is that if X is contained in a hypersurface of degree d , then $MS_k(X) \subset X$ for all $k > d$ (and thus is empty if X contains no lines). See [R2] for one use of this observation.

2.3 Tangential varieties.

If X is smooth, the tangential variety of X , $\tau(X) \subset \mathbb{P}V$, is simply the union of all the embedded tangent spaces. When X is not smooth, there are several possible notions one could use to define tangent spaces (see [L5..]). The notion that turns out to be useful is the *tangent star*, $T_x^*X \subset \mathbb{P}V$. Intuitively, T_x^*X is the limit of secant lines. More precisely, let $x \in X$, \mathbb{P}_*^1 is a line in T_x^*X if there exist smooth curves $p(t), q(t)$ on X such that $p(0) = q(0) = x$ and $\mathbb{P}_*^1 = \lim_{t \rightarrow 0} \mathbb{P}_{pq}^1$. T_x^*X is the union of all \mathbb{P}_*^1 's at x and we define the *tangential variety of X* , by $\tau(X) = \bigcup_{x \in X} T_x^*X$. One can define higher tangential varieties to be the union of higher osculating spaces, i.e., $\tau_k(X) := \bigcup T_x^{\star(k)}X$, where at smooth points $T_x^{\star(k)}X = \tilde{T}_x^{(k)}X$ is the usual osculating space and at singular points one takes the union of \mathbb{P}^{k-1} 's that are limits of k points on X moving towards x . Tangential varieties are sometimes called *developpable varieties*.

§3. TOPOLOGY AND CONSEQUENCES

Three basic theorems about the topology of subvarieties of projective space are Bertini's theorem, Lefshetz's theorem and Bezout's theorem.

Bezout's theorem states that varieties of complementary dimension must intersect (generalizing the fundamental theorem of algebra). We have already seen that Bezout's theorem has consequences for the infinitesimal geometry of subvarieties. We will see the same is true of the Bertini and Lefschetz theorems. The Bertini and Lefschetz theorems and their variants are also quite useful at the infinitesimal level to prove results about systems of quadrics.

3.1 Bertini's Theorem. *Let $A \subset S^d V^*$ be a linear subspace. Let $P \in A$ be a general element and let $Z_P \subset \mathbb{P}V$ be the hypersurface of degree d that it determines. Then $(Z_P)_{\text{sing}} \subseteq \text{Base}(A)$.*

Bertini's theorem can be understood as a quantitative version of Sard's theorem for polynomials. Its proof is elementary, see e.g. [GH2].

One of the most important topological theorems in algebraic geometry is the Lefschetz theorem. Its simplest form is:

3.2 Lefschetz theorem, version 1. *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety and let $H \subset \mathbb{P}^{n+a}$ be a hyperplane, then the restriction map on cohomology:*

$$H^i(X, \mathbb{Z}) \rightarrow H^i(X \cap H, \mathbb{Z})$$

is an isomorphism for $i < n - 1$ and injective for $i = n - 1$.

There are two standard proofs of the Lefschetz theorem, one using Morse theory (see [Milnor]), and the other using harmonic differential forms and the Hodge theorem (see [GH2]). In both proofs, an essential point is that when one writes a $n \times n$ Hermitian matrix with complex entries as a $2n \times 2n$ matrix with real entries, the eigenvalues of the new matrix will occur in pairs $\lambda, -\lambda$.

Note that if X is smooth, then by Poincaré duality the cohomology of $X \cap H$ is determined in all but the three middle dimensions.

Now let $Z \subset \mathbb{P}^{n+a}$ be a smooth hypersurface of degree d and consider $X \cap Z$. By re-embedding \mathbb{P}^{n+a} by the d -th Veronese, we may linearize the equation of Z , so the same conclusion holds for the map $H^i(X, \mathbb{Z}) \rightarrow H^i(X \cap Z, \mathbb{Z})$. Call this extension version 2.

If we take $X = \mathbb{P}^{n+a}$ and cut by hypersurfaces, as long as we have a smooth variety, i.e. a smooth complete intersection at each step, we can continue iteratively to get:

3.3 Lefschetz theorem version 3. *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth complete intersection. Then the restriction map on cohomology:*

$$H^i(\mathbb{P}^{n+a}, \mathbb{Z}) \rightarrow H^i(X, \mathbb{Z})$$

is an isomorphism for $i \neq n$ and injective for $i = n$.

Thus for smooth complete intersections, almost all the cohomology is inherited from the ambient projective space. The Lefschetz theorem extends to fundamental groups, see, e.g. [Ful], so a good deal of the topology of a complete intersection is inherited from the ambient projective space.

The hypotheses on the Lefschetz theorem can be relaxed in several different ways. Here is a generalization due to Goresky and MacPherson:

3.4 Theorem [GM]. *Let X^n be an algebraic variety and suppose $\pi : X \rightarrow \mathbb{P}^{n+a}$ is a (not necessarily proper) algebraic map with finite fibers. Let $L \subset \mathbb{P}^{n+a}$ be a linear subspace of codimension c . Let L_ϵ denote an ϵ -neighborhood of L with respect to some Riemannian metric on \mathbb{P}^{n+a} .*

If $\epsilon > 0$ is sufficiently small, then the inclusion $\pi^{-1}(L_\epsilon) \rightarrow X$ induces an isomorphism on intersection homology groups $IH_i(\pi^{-1}(L_\epsilon), \mathbb{Z}) \simeq IH_i(X, \mathbb{Z})$ for all $i < n - c$ and a surjection $IH_{n-c}(\pi^{-1}(L_\epsilon), \mathbb{Z}) \rightarrow IH_{n-c}(X, \mathbb{Z})$.

Furthermore, if L is generic, then L_ϵ may be replaced by L in the above formulae.

Using the generalized version of the Lefschetz theorem (actually other results that are essentially equivalent, e.g. Deligne's extension of Bertini's theorem) Fulton and Hansen proved the following theorem which has striking consequences:

3.5 Connectedness Theorem [FH]. *Let Z^n be a projective variety, let $f : Z \rightarrow \mathbb{P}^m \times \mathbb{P}^m$ be a finite morphism, and let $\Delta \subset \mathbb{P}^m \times \mathbb{P}^m$ denote the diagonal. If $n > m$, then $f^{-1}(\Delta)$ is connected.*

As explained by Fulton, see [Ful], Bertini, Bezout and Lefschetz's theorems can all be understood in terms of connectedness.

An important application of the connectedness theorem is the following, which reduces the studies of degenerate secant and tangential varieties to the same problem.

If $Y \subseteq X \subseteq \mathbb{P}^{n+a}$, define $\tau(Y, X) = \cup_{y \in Y} T_y^* X$.

3.6 Theorem [Z], [FH]. *Let $X^n, Y^y \subset \mathbb{P}V$ be varieties, respectively of dimensions n, y . Assume $Y \subseteq X$. Then either*

$$\dim \sigma(Y, X) = n + y + 1 \text{ and } \dim \tau(Y, X) = n + y$$

or

$$\sigma(Y, X) = \tau(Y, X).$$

Taking $Y = X$, one sees that if either $\sigma(X)$ or $\tau(X)$ is not of the expected dimension, then they must be equal, which is the version proved by Fulton and Hansen. The extension is due to Zak.

Proof. Assume $\dim \tau(Y, X) = t < n + y$. We need to show $\sigma(Y, X) = \tau(Y, X)$. Project \mathbb{P}^{n+a} from some linear space $L^{n+a-t-1}$ avoiding $\tau(Y, X)$ to a \mathbb{P}^t and consider the map $F : X \times Y \rightarrow \mathbb{P}^t \times \mathbb{P}^t$. Since F restricted to each factor is finite, $\dim F(X \times Y) = n + y > t$, so $F^{-1}(\Delta)$ is connected. Assume $\sigma(Y, X) \neq \tau(Y, X)$, so $\dim \sigma(Y, X) > \tau(Y, X)$ and thus $\sigma(Y, X)$ intersects L . Thus there exists $x \in X \setminus Y$, $y \in Y$, such that the line $\overline{xy} \cap L \neq \emptyset$. Thus $F(x, y) = F(y, y)$. By the connectedness theorem, there exists an arc (x_t, y_t) in $F^{-1}(\Delta)$ such that $(x_0, y_0) = (x, y)$ and $(x_1, y_1) = (y, y)$. By continuity there exists some minimal t' such that $x_{t'} = y_{t'}$ and thus the limiting line as $t \rightarrow t'$ is a tangent line that intersects L , giving a contradiction. \square

The following important application of the connectedness theorem is due to Zak:

3.7 Zak's theorem on tangencies, [Z]. *Let $b = \dim X_{\text{sing}}$ (Set $b = -1$ if X is smooth.) Let $L \in \gamma_k(X)$ be any point, then $\dim \{x \in X \mid \tilde{T}_x X \subseteq L\} \leq k + (b + 1)$.*

proof. Assume X is smooth. Let $Y = Y_L$ and let $y = \dim Y$. We have $\tau(Y, X) \subset L$, but since X is not contained in a hyperplane $\sigma(Y, X) \not\subseteq L$. Thus by (3.6), $\dim \tau(Y, X) = y + n$. Thus $y + n \leq n + k$, i.e. $y \leq k$. \square

3.7.1 Exercise. Prove the general case. Hint: consider the intersection of X with a linear space.

In comparison to Lefschetz theorem version 3, Barth and Larsen proved the following theorem about an arbitrary smooth subvariety of projective space:

3.8 Theorem, see [B], [BL], [Hart]. *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety, then the restriction map on cohomology*

$$H^i(\mathbb{P}^{n+a}, \mathbb{Z}) \rightarrow H^i(X, \mathbb{Z})$$

is an isomorphism for $i \leq n - a$.

Thus, smooth varieties of small codimension cohomologically “look like” complete intersections.

A basic question in geometry is: To what extent is the geometry of a manifold determined by its topology? For example, in Riemannian geometry, certain averages of the Riemann curvature tensor are invariants of the differentiable topology.

R. Hartshorne posed several questions in algebraic geometry to the effect of asking when something is true on the level of cohomology implies the corresponding result on the level of geometry. For example, there is the classical question: under what circumstances are cohomology classes represented by an algebraic cycle (or variety)? Among his questions is the following famous conjecture:

3.9 Hartshorne's conjecture on complete intersections. ([Hr], 1974) Let $X^n \subset \mathbb{CP}^{n+a}$ be a smooth variety. If $a < \frac{n}{2}$ then X is a complete intersection.

If X is a complete intersection, then all polynomials of degree d on X are the restriction of polynomials of degree d on $\mathbb{P}V$. In other words, the map $H^0(\mathbb{P}^{n+a}, \mathcal{O}_{\mathbb{P}^{n+a}}(d)) \rightarrow H^0(X, \mathcal{O}_X(d))$ is a surjection. If all up to the d -th maps are surjective, one says that X is *d-normal*, and if all maps are surjective, one says that X is *projectively normal*. 1-normal varieties are called *linearly normal*. Another way of saying that X is linearly normal is that X cannot be realized as the linear projection of some $\tilde{X} \subset \mathbb{P}^{n+a+1}$. More generally, X is *d-normal* if $v_d(X)$ is not the linear projection of some $Y \subset \mathbb{P}^{\binom{n+a+d}{d}}$. It has been shown that under certain circumstances, if X is projectively normal, it must be a complete intersection **ref?**. Thus we may think of X being *d* normal as an approximation to being a complete intersection.

Thus a first approximation to Hartshorne's conjecture on complete intersections (also conjectured by Hartshorne) is that if $X^n \subset \mathbb{P}^{n+a}$ is a smooth variety, it must be linearly normal if $a < \frac{n}{2}$. Now say $Y \subset \mathbb{P}^{N+1}$ is a smooth variety. We can project Y to a smooth subvariety of some \mathbb{P}^N iff there exists a $p \in \mathbb{P}^{N+1}$ such that p does not lie on any secant or tangent line of Y , i.e. iff $\sigma(Y) \neq \mathbb{P}^{N+1}$. Zak proved that if $N+1-n \leq \frac{n}{2}$ then $\sigma(Y) = \mathbb{P}^{N+1}$ and this is his theorem on linear normality quoted in the introduction.

§4. PROJECTIVE DIFFERENTIAL INVARIANTS

4.1 The moving frame.

As discussed in the introduction, we will study the local geometry of a subvariety $X^n \subset \mathbb{P}V$ by studying the infinitesimal motions of flags in V , in fact of bases of V adapted to the geometry of X .

Let $N = n+a$. To begin, let \mathcal{F} denote the space of bases of V . \mathcal{F} is isomorphic to $GL(V)$. We write $f \in \mathcal{F}$ as $f = (e_0, \dots, e_N)$, where we think of the basis vectors $e_B \in V$ as column vectors.

We will use the projection

$$(4.1.1) \quad \begin{aligned} \mathcal{F} &\rightarrow \mathbb{P}V \\ f &= (e_0, \dots, e_N) \mapsto [e_0]. \end{aligned}$$

4.1.2 Exercise. Define a projection $\mathcal{F} \rightarrow \mathbb{G}(k, \mathbb{P}V)$. What is the group preserving the fiber?

We will often consider $f \in \mathcal{F}$ as a matrix, e.g. $f = (g_B^A)$, writing

$$e_B = \begin{pmatrix} g_B^0 \\ \vdots \\ g_B^N \end{pmatrix}.$$

On any Lie group G , there is a canonical left-invariant \mathfrak{g} -valued one-form, the *Maurer-Cartan form* $\Omega \in \Omega^1(G, \mathfrak{g})$.

In the case of a matrix Lie group (our situation), we think of the matrix elements g_B^A as coordinate functions, and take their derivative to get a mapping $dg_f : T_f \mathcal{F} \rightarrow$

$M_{N+1 \times N+1}(\mathbb{C})$. ($M_{k \times l}(\mathbb{C})$ denotes the vector space of $k \times l$ matrices.) In this situation, the Maurer-Cartan form is

$$\Omega := g^{-1}dg \in \Omega^1(\mathcal{F}, M_{N+1 \times N+1}(\mathbb{C})).$$

Utilizing the fact that the differential of a constant map is zero, we calculate $0 = d(gg^{-1}) = gdg^{-1} + (dg)g^{-1}$ and find that $d\Omega = d(g^{-1}dg) = d(g^{-1}) \wedge dg = -g^{-1}dgg^{-1} \wedge dg$. I.e.

$$(4.1.3) \quad d\Omega = -\Omega \wedge \Omega,$$

which is called the *Maurer-Cartan equation*. In indicies,

$$(4.1.4) \quad d\omega_B^A = -\omega_C^A \wedge \omega_B^C$$

(here and throughout, we use the convention that repeated indicies occuring up and down are to be summed over).

ω_B^A has the geometric interpretation of measuring the infinitesimal motion of e_A towards e_B , as $de_A = \omega_A^0 e_0 + \dots + \omega_A^N e_N$. In particular, the infinitesimal motion of e_0 is measured by the forms ω_0^B , so it is not suprising that $\{\omega_0^1, \dots, \omega_0^N\} = \pi^*(T_{[e_0]} \mathbb{P}V)$.

We now adapt to the geometry of our situation, the flag $\hat{x} \subset \hat{T}_x X \subset V$. We let $\mathcal{F}_X^1 = \mathcal{F}^1 \subset \mathcal{F}$ be the subbundle respecting this flag.

Write $f = (e_0, e_\alpha, e_\mu)$ for an element of \mathcal{F}^1 , where $1 \leq \alpha, \beta \leq n$, $n+1 \leq \mu, \nu \leq n+a$ where $[e_0] = x$, and $\{e_0, \dots, e_n\} = \hat{T}$. \mathcal{F}^1 is not a Lie group, but it is a G_1 -principal bundle where

$$(4.1.5) \quad G_1 = \left\{ g \in GL(V) \mid g = \begin{pmatrix} g_0^0 & g_\beta^0 & g_\nu^0 \\ 0 & g_\beta^\alpha & g_\nu^\alpha \\ 0 & 0 & g_\nu^\mu \end{pmatrix} \right\}$$

Let $i : \mathcal{F}^1 \subset \mathcal{F}$ denote the inclusion.

$i^*(\omega_0^\mu) = 0$ because $de_0 \equiv \omega_0^\alpha e_\alpha \bmod e_0$ and for the same reason, $i^*(\omega_0^1 \wedge \dots \wedge \omega_0^n)$ is nonvanishing.

From now on, I commit a standard abuse of notation, omitting the i^* in the notation, the pullback being understood from the context.

Expanding out $0 = d\omega_0^\mu = -\omega_\beta^\mu \wedge \omega_0^\beta$ using the Maurer-Cartan equation (4.1.4) yields

$$(4.1.6) \quad \omega_\beta^\mu = q_{\alpha\beta}^\mu \omega_0^\alpha$$

for some functions $q_{\alpha\beta}^\mu = q_{\beta\alpha}^\mu$ defined on \mathcal{F}^1 .

To prove the symmetry, one uses the *Cartan lemma*: Let W be a vector space and let w_1, \dots, w_k be independent elements. If $v_1, \dots, v_k \in W$ are such that $w_1 \wedge v_1 + \dots + w_k \wedge v_k = 0$, then $v_j = \sum_i h_{ij} w_i$ with $h_{ij} = h_{ji}$.

4.1.7 Exercise. prove the Cartan Lemma.

Consider

$$(4.1.8) \quad \tilde{II} = q_{\alpha\beta}^\mu \omega_0^\alpha \omega_0^\beta \otimes (e_\mu \bmod \hat{T})$$

4.1.9 Exercises.

1. Show that $\tilde{II} \otimes e_0^*$ is invariant under motions in the fiber over $[e_0]$ and thus descends to be a well defined section of $S^2 T^* X \otimes N X$, which, as expected, is II .
2. Show that this definition is equivalent to the definitions of II given in the introduction. Hint: use the projection $\mathcal{F}^1 \rightarrow G(k+1, V)$ to show equivalence with the Gauss map definition and use a section of \mathcal{F}^1 given in coordinates to prove the second equivalence, e.g., take $e_\alpha = \frac{\partial}{\partial x^\alpha} - f_{x^\alpha}^\mu \frac{\partial}{\partial x^\mu}$, $e_\mu = \frac{\partial}{\partial x^\mu}$.

4.2 Higher fundamental forms.

We can define the higher fundamental forms \mathbb{FF}^k as maps $\mathbb{FF}^k : \ker \mathbb{FF}^{k-1} \rightarrow S^k T^*$, where we begin by considering $II = \mathbb{FF}^2 : N^* \rightarrow S^2 T^*$. Note that in projective geometry this mapping is more natural than the dual map $S^2 T \rightarrow N$ as $\mathbb{P} N_x^* X$ has the geometric interpretation of the space of hyperplanes tangent to X at x .

In coordinates, the k -th fundamental form corresponds to the first nonzero terms in the Taylor series expansion using coordinates adapted to the filtration of N^* .

Fix a general point $x \in X$ and let $a_1 = \dim II(T, T)$ and let

$$\hat{T}_x^{(2)} X = \hat{T} + II(T, T)(-1) \subseteq V$$

denote the *second osculating space* of X at x .

One way to define III is as the derivative of the second Gauss map

$$(4.2.1) \quad \begin{aligned} \gamma^{(2)} : X &\dashrightarrow G(n + a_1, V) \\ x &\mapsto \hat{T}^{(2)} \end{aligned}$$

4.2.2 Exercise.

Write out this definition.

To define \mathbb{FF}^3 in frames, consider the quantity

$$(4.2.3) \quad \tilde{\mathbb{FF}}^3 := \omega_\mu^\nu \omega_\beta^\mu \omega_0^\beta \otimes e_\nu \text{mod } \hat{T}^{(2)}$$

$\tilde{\mathbb{FF}}^3$ descends (after twisting) to a well defined element $\mathbb{FF}^3 \in \Gamma(S^3 T^* \otimes (N/II(S^2 T)))$. \mathbb{FF}^3 has the geometric interpretation of measuring how X is moving away from its second osculating space to first order. We sometimes use III to denote \mathbb{FF}^3 .

Adapt frames to the filtration of N , letting $\{e_\xi\} = II(S^2 T)$ and $\{e_\phi\} = N \text{ mod } II(S^2 T)$, with index ranges $n+1 \leq \xi, \eta \leq n+a_1$, $n+a_1+1 \leq \phi, \psi \leq n+a$. Let \mathcal{F}^2 denote the resulting bundle. Then

$$(4.2.4) \quad \tilde{\mathbb{FF}}^3 := \omega_\xi^\phi \omega_\beta^\xi \omega_0^\beta \otimes e_\phi \text{mod } \hat{T}^{(2)}.$$

4.2.5 Exercise.

Using this definition, show that \mathbb{FF}^3 is a symmetric cubic form.

There is a further restriction on \mathbb{FF}^3 at general points. (4.2.3) implies so $|\mathbb{FF}^3| \subseteq (|II| \otimes T^*)$.

4.2.6 Definition. Let T be a vector space and let $A \subset S^d T^*$ be a linear subspace. Define the k -th *prolongation* of A by $A^{(k)} := S^{d+k} T^* \cap (A \otimes S^k T^*)$. Geometrically, $A^{(k)}$ is the space of polynomials of degree $d+k$ on T having the property that all their k -th derivatives lie in A .

4.2.7 Theorem (Cartan) [C vIII.1 p 377]. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and let $x \in X$ be a general point. Then*

$$(4.2.8) \quad |\mathbb{FF}^k|_{X,x} \subseteq |II|_{X,x}^{(k-1)}.$$

We will call (4.2.8) the *prolongation property*. It can fail to hold at special points where II degenerates.

The two remarks above prove Cartan's theorem for \mathbb{FF}^3 , the general case is left as an exercise.

4.3 Relative differential invariants.

You may wish to skip this section on a first reading. Life would be easier if one could just skip it altogether, but unfortunately the cubic form plays an important role any time the geometry of X is not completely determined by $|II|$.

In addition to III , there is another third order invariant called the *cubic form*, which we will denote F_3 . The geometric interpretation of F_3 is that it measures how the second fundamental form varies infinitesimally, or in other words, how X is leaving its embedded tangent space to second order.

Namely, differentiating $0 = d(\omega_\alpha^\mu - q_{\alpha\beta}^\mu \omega_0^\beta)$ implies there exist functions $r_{\alpha\beta\gamma}^\mu$ defined on \mathcal{F}^1 that satisfy

$$(4.3.1) \quad r_{\alpha\beta\gamma}^\mu \omega_0^\gamma = -dq_{\alpha\beta}^\mu - q_{\alpha\beta}^\mu \omega_0^0 - q_{\alpha\beta}^\nu \omega_\nu^\mu + q_{\alpha\delta}^\mu \omega_\beta^\delta + q_{\beta\delta}^\mu \omega_\alpha^\delta.$$

For $f \in \mathcal{F}^1$, let $F_3 = (F_3)_f \in \pi^*(S^3 T^* \otimes N)$ be

$$(4.3.2) \quad F_3 = r_{\alpha\beta\gamma}^\mu \omega_0^\alpha \omega_0^\beta \omega_0^\gamma \otimes e_\mu.$$

Note that F_3 does *not* descend to be a well defined section of $S^3 T^* \otimes N$ (even after twisting). In fact, if $(\tilde{e}_0, \tilde{e}_\alpha, \tilde{e}_\mu)$ is a new frame with

$$(4.3.3) \quad \begin{aligned} \tilde{e}_\mu &= e_\mu + g_\mu^0 e_0 + g_\mu^\alpha e_\alpha \\ \tilde{e}_\alpha &= e_\alpha + g_\alpha^0 e_0 \end{aligned}$$

then

$$(4.3.4) \quad \tilde{r}_{\alpha\beta\gamma}^\mu = r_{\alpha\beta\gamma}^\mu + \mathfrak{S}_{\alpha\beta\gamma} g_\alpha^0 q_{\beta\gamma}^\mu + \mathfrak{S}_{\alpha\beta\gamma} g_\nu^\delta q_{\alpha\beta}^\nu q_{\gamma\delta}^\mu$$

Motions by $g_\nu^\mu, g_\beta^\alpha, g_0^0$ also vary the $r_{\alpha\beta\gamma}^\mu$ but the change they effect is cancelled by the corresponding changes in the ω_0^α and e_μ . F_3 is an example of what is called a *relative invariant*. We will use the notation $\Delta r_{\alpha\beta\gamma}^\mu$ to denote the change in $r_{\alpha\beta\gamma}^\mu$ by a fiber motion of the type in (4.3.3). By (4.3.4),

$$(4.3.5) \quad \Delta r_{\alpha\beta\gamma}^\mu = \mathfrak{S}_{\alpha\beta\gamma} (g_\alpha^0 q_{\beta\gamma}^\mu + g_\nu^\delta q_{\alpha\beta}^\nu q_{\gamma\delta}^\mu).$$

It is possible to define F_3 as a section of a bundle well defined over X , namely

$$(4.3.6) \quad F_3 \in \Gamma(X, \frac{S^3 T^* \otimes N}{T^* \circ II + \langle II, T \otimes N^*, II \rangle})$$

where $\langle \cdot, \cdot, \cdot \rangle$ is the natural contraction.

One can compare F_3 to the covariant derivative of the second fundamental form of a submanifold of a Riemannian manifold, $\nabla^{riem} II$, which is a well defined tensor. In fact one may think of F_3 as an the equivalence class of $\nabla^{riem} II$'s, where they range over the (holomorphic) metrics compatible with the projective structure.

Differentiating F_3 one obtains a fourth order invariant $F_4 \in \pi^*(S^4 T^* \otimes N)$ whose coefficients $r_{\alpha\beta\gamma\delta}^\mu$ are defined by

$$(4.3.7) \quad r_{\alpha\beta\gamma\delta}^\mu \omega_0^\delta = -dr_{\alpha\beta\gamma}^\mu - r_{\alpha\beta\gamma}^\mu \omega_0^0 - r_{\alpha\beta\gamma}^\nu \omega_\nu^\mu + \mathfrak{S}_{\alpha\beta\gamma}(r_{\alpha\beta\epsilon}^\mu \omega_\gamma^\epsilon - q_{\alpha\beta}^\mu \omega_\gamma^0 + q_{\alpha\epsilon}^\mu q_{\beta\gamma}^\nu \omega_\nu^\epsilon).$$

(If $III = 0$, F_4 is the only fourth order invariant.) The geometric interpretation of F_4 is that it measures how X leaves its embedded tangent space to third order.

Under a change of frame (4.3.4) the coeffecients of F_4 vary as follows:

$$(4.3.8) \quad \tilde{r}_{\alpha\beta\gamma\delta}^\mu = r_{\alpha\beta\gamma\delta}^\mu + \mathfrak{S}_{\alpha\beta\gamma\delta} g_\alpha^0 r_{\beta\gamma\delta}^\mu + \mathfrak{S}_{\alpha\beta\gamma\delta} g_\nu^\epsilon (r_{\alpha\beta\gamma}^\nu q_{\delta\epsilon}^\mu + q_{\alpha\beta}^\nu r_{\gamma\delta\epsilon}^\mu) + g_\nu^0 q_{\alpha\beta}^\mu q_{\gamma\delta}^\nu.$$

One can continue, defining forms F_k for all k .

In coordinates adapted to a point x , the coefficients of the F_k at x are the k -th derivatives of the embedding.

4.4 Yet another definition of fundamental forms.

Here is yet another definition of the fundamental forms that will be particularly useful for calculating fundamental forms of re-embeddings and homogeneous spaces. If you are more at home with spectral sequences than Gauss maps, coordinates, or moving frames, then this definition is for you.

Define inductively a series of maps (following [L6]):

$$(4.4.1) \quad \underline{d}^k e_0 : (T\mathcal{F}^1)^{\otimes k} \rightarrow V/\text{Image}(\underline{d}^0, \dots, \underline{d}^{k-1})$$

as follows: Let d denote exterior differentiation, let $\underline{d}^0 e_0 = e_0$ and let $\underline{d}^1 e_0 = de_0 \bmod e_0$. If $v_1, \dots, v_k \in T_f \mathcal{F}^1$, extend v_1, \dots, v_k to holomorphic vector fields in some neighborhood of f which we denote $\tilde{v}_1, \dots, \tilde{v}_k$. Let

$$(4.4.2) \quad \underline{d}^k e_0(v_1, \dots, v_k) := v_1(\lrcorner d(\tilde{v}_2 \lrcorner \dots \lrcorner d(\tilde{v}_k \lrcorner de_0) \bmod \pi_k^{-1}(\text{Image} \underline{d}^{k-1}))$$

where $\pi_k : V \rightarrow V/(\text{Image}\{\underline{d}^0, \dots, \underline{d}^{k-1}\})$ is the projection, and \lrcorner denotes the contraction $T \times T^{*\otimes l} \rightarrow T^{*\otimes l-1}$. (4.4.2) is independent of the extension of v_1, \dots, v_k to holomorphic vector fields. (The proof that (4.4.2) is independent of the choice of extension to holomorphic vector fields is the same as the standard argument in the real case, see e.g. [S].) $\underline{d}^k e_0 \otimes (e_0^*)^{k-1} = \pi^*(\mathbb{F}^k)$. The maps \underline{d}^k can be defined more algebraically as follows;

The quotient map

$$(4.4.3) \quad V^* \rightarrow V^*/\hat{x}^\perp = \mathcal{O}_{\mathbb{P}V}(1)_x$$

gives rise to a spectral sequence of a filtered complex by letting

$$(4.4.4) \quad \begin{aligned} F^0 K^0 &= V^* & F^0 K^1 &= \mathcal{O}_X(1)_x \\ F^1 K^0 &= 0 & F^p &= F^p K^1 = \mathfrak{m}_x^p(1). \end{aligned}$$

The maps are

$$\begin{aligned}
 (4.4.5) \quad \underline{d}^0 : V^* &\rightarrow F^0/F^1 = \mathcal{O}_{X,x}(1)/\mathfrak{m}_x(1) \simeq \mathbb{C} \\
 \underline{d}^1 : \ker \underline{d}^0 &\rightarrow F^1/F^2 = \mathfrak{m}_x(1)/\mathfrak{m}_x^2(1) \simeq T^*(1) \\
 \underline{d}^2 : \ker \underline{d}^1 &\rightarrow F^2/F^3 = \mathfrak{m}_x^2(1)/\mathfrak{m}_x^3(1) \simeq (S^2 T^*)(1) \\
 &\vdots
 \end{aligned}$$

For example, the first two terms of (4.4.5) expressed in frames are

$$(4.4.6) \quad \underline{d}^1 e_0 = \omega_0^\alpha \otimes e_\alpha \bmod \hat{x}$$

$$(4.4.7) \quad \underline{d}^2 e_0 = \omega_0^\alpha \omega_\alpha^\mu \otimes e_\mu \bmod \hat{T}$$

We will use the notation $F^1 = \mathbb{F}\mathbb{F}^1 = Id_T$, $F^0 = \mathbb{F}\mathbb{F}^0 = \hat{x} \otimes \hat{x}^*$.

This definition is particularly useful in computing fundamental forms of homogeneous spaces. Say $G/P \subset \mathbb{P}V$ and V is a vector space formed from a vector space W (e.g. an exterior power) and originally $G \subseteq GL(W) \subset GL(V)$. Then we can use the smaller G -frame bundle to do our computations.

4.5 Fundamental forms of Veroneses $v_p(\mathbb{P}V) \subset \mathbb{P}S^p V$.

Let $V = \mathbb{C}^{n+1}$ have basis $\{e_0, e_\alpha\}$, $1 \leq \alpha \leq n$ and let $x = [(e_0)^p]$. Using the bundle of $\rho_p(GL(V)) \subset GL(S^p V)$ frames, we have

$$\begin{aligned}
 (4.5.1) \quad \underline{d}e_0^d &\equiv p\omega_0^\alpha e_\alpha e_0^{p-1} \\
 \underline{d}^2 e_0^d &\equiv p(p-1)\omega_0^\alpha \omega_0^\beta e_\alpha e_\beta e_0^{p-2} \\
 \underline{d}^k e_0^d &\equiv p(p-1) \dots (p-k+1) \omega_0^{\alpha_1} \dots \omega_0^{\alpha_k} e_{\alpha_1} \dots e_{\alpha_k} e_0^{p-k}
 \end{aligned}$$

Thus

$$\begin{aligned}
 (4.5.2) \quad |\mathbb{F}\mathbb{F}_{v_p(\mathbb{P}V)}^k| &= \mathbb{P}S^k T^* \quad k \leq p \\
 |\mathbb{F}\mathbb{F}_{v_p(\mathbb{P}V)}^k| &= \emptyset \quad k > p.
 \end{aligned}$$

4.6 Fundamental forms of Veronese re-embeddings $v_d(X) \subset \mathbb{P}S^d V$.

4.6.1 Fundamental forms of $v_2(X) \subset \mathbb{P}S^2 V$. Assume $X \subset \mathbb{P}V$ is such that II_X is surjective so the only differential invariants are $F_k = (F_k)_{X,x}$. Write $x = [e_0]$, $v_2(x) = [e_0 \circ e_0]$ and use the Leibnitz rule applied to $e_0 \circ e_0$ to compute (see [L6, 3.2] for details):

$$\begin{aligned}
 (4.6.2) \quad \mathbb{F}\mathbb{F}_{v_2(X),x}^1 &= 2F_1 F_0|_{\hat{x}^2 \perp} \\
 \mathbb{F}\mathbb{F}_{v_2(X),x}^2 &= 2(F_2 F_0 + F_1 F_1)|_{(\hat{x} \hat{T})^\perp} \\
 \mathbb{F}\mathbb{F}_{v_2(X),x}^3 &= 2(F_3 F_0 + 3F_2 F_1)|_{\ker \mathbb{F}\mathbb{F}_{v_2(X)}^2} \\
 \mathbb{F}\mathbb{F}_{v_2(X),x}^4 &= 2(F_4 F_0 + 4F_3 F_1 + 3F_2 F_2)|_{\ker \mathbb{F}\mathbb{F}_{v_2(X)}^3} \\
 \mathbb{F}\mathbb{F}_{v_2(X),x}^5 &= 2(F_5 F_0 + 5F_4 F_1 + 10F_3 F_2)|_{\ker \mathbb{F}\mathbb{F}_{v_2(X)}^4} \\
 &\vdots
 \end{aligned}$$

4.6.3 Proposition [L6, 3.10]. *The fundamental forms of $v_d(X)$ are*

$$(4.6.4) \quad \mathbb{FF}_{v_d(X)}^k = \sum_{l_1+\dots+l_d=k} c_{l_1\dots l_d} F_{l_1} \dots F_{l_d} \bmod (\sum_{l < k} \mathbb{FF}_{v_d(X)}^l(S^l T))|_{\ker \mathbb{FF}_{v_d(X)}^{k-1}}$$

where the $c_{l_1\dots l_d}$ are nonzero constants.

For example

$$(4.6.5) \quad \mathbb{FF}_{v_3(X)}^4 = c_{400} F_4 F_0 F_0 + c_{310} F_3 F_1 F_0 + c_{220} F_2 F_2 F_0 + c_{211} F_2 F_1 F_1.$$

For the proof, see [L6].

Note that if Z is a hypersurface of degree d , then Z osculates to order p at $x \in X$ if and only if $H_Z \in \ker \mathbb{FF}_{v_d(X), x^d}^k$.

4.6.4 Proofs of (0.5.8), (0.5.9).

Proof of [L6, 3.16]. Observe that $|\mathbb{FF}_{v_d(X)}^k| = \mathbb{P}S^k T^*$ for all $k \leq d$ and sum up. \square

Proof of [L6, 3.17]. The first term in any $\mathbb{FF}_{v_d(X)}^k$ for which $S^d N^*$ is not in the kernel is $(F_2)^d$, which appears in $\mathbb{FF}_{v_d(X)}^{2d}$. Thus $S^d N^* \not\subseteq \ker \mathbb{FF}_{v_d(X)}^{2d-1}$. Finally, note that $\dim S^d N^* = \binom{a+d-1}{d}$. \square

4.7 Fundamental forms of Grassmannians.

Let $W = \mathbb{C}^n$ and let $G(k, W) \subset \mathbb{P}(\Lambda^k W)$. Write $V = \Lambda^k W$. Use index ranges $1 \leq i, j \leq k$, $k+1 \leq s, t \leq n$. Write the Maurer-Cartan form of $GL(W)$ as

$$(4.7.1) \quad \Omega = \begin{pmatrix} \omega_j^i & \omega_t^i \\ \omega_j^s & \omega_t^s \end{pmatrix}$$

Using the embedding $\rho : GL(W) \rightarrow GL(V)$, let $E = e_1 \wedge \dots \wedge e_k \in G(k, W)$, $E_s^j = e_1 \wedge \dots \wedge e_{j-1} \wedge e_s \wedge e_{j+1} \wedge \dots \wedge e_k$ where e_j has been replaced by e_s , let E_{st}^{ij} be E with e_s replacing e_i and e_t replacing e_j and so on. Then the $E_{s_1\dots s_p}^{j_1\dots j_p}$, $1 \leq p \leq k$, give a basis of V , in fact adapted to the representation ρ . Now we take derivatives at E using (4.4) and the Leibnitz rule:

$$(4.7.2) \quad \begin{aligned} \underline{d}E &\equiv \omega_i^s E_s^i \bmod E \\ \underline{d}^2 E &\equiv \sum_{i < j, s < t} (\omega_i^s \omega_j^t - \omega_i^t \omega_j^s) E_{st}^{ij} \\ \underline{d}^3 E &\equiv \sum_{i < j < l, s < t < u} \det \begin{pmatrix} \omega_i^s & \omega_i^t & \omega_i^u \\ \omega_j^s & \omega_j^t & \omega_j^u \\ \omega_l^s & \omega_l^t & \omega_l^u \end{pmatrix} E_{stu}^{ijl} \end{aligned}$$

The first line can be used to recover that $T_E G(k, W) = E^* \otimes W/E$. Continuing, we see that for any $x \in G(k, W)$,

$$(4.7.3) \quad |\mathbb{FF}^p| = \{p \times p \text{ minors of } (\omega_j^s)\} = I_p \sigma_{p-1}(\text{Seg}(\mathbb{P}E^* \times \mathbb{P}(V/E))).$$

In particular, the last nonzero fundamental form is the $\min(k, n-k)$ -th.

4.8 Fundamental forms of spinor varieties.

Choose a basis of $V = \mathbb{C}^{2m}$ so that $Q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. With respect to this basis $SO(V, Q)$ has Maurer-Cartan form

$$(4.8.1) \quad \begin{pmatrix} \omega_j^i & \omega_{n+k}^i \\ \omega_{n+l}^i & \omega_{n+l}^i \end{pmatrix}$$

where $\omega_{n+l}^{n+k} = \omega_l^k$, $\omega_{n+k}^i = -\omega_{n+i}^k$, $\omega_j^{n+l} = -\omega_l^{n+j}$.

Let $E = e_1 \wedge \dots \wedge e_m \in \mathbb{S}_m = \mathbb{S}_m(V, Q)$. Computing as for the Grassmannians,

$$(4.8.2) \quad \begin{aligned} dE &\equiv \omega_j^{n+i} E_{n+i}^j \bmod E \\ d^2E &\equiv \omega_j^{n+i} \omega_l^{n+k} E_{n+i, n+k}^{jl} \bmod \text{Image } dE \\ &\vdots \end{aligned}$$

Note that Q allows us to identify V/E with E^* . The additional structure on the tangent space induced from that of the Grassmannian becomes

$$T = \Lambda^2 E^*.$$

Moreover,

$$|\mathbb{FF}^k| \simeq \Lambda^{2k} E^* = I_k \sigma_{k-1} G(2, E)$$

in particular $|II| = I_2 G(2, E)$.

Note that one can construct the space of positive spinors $\Lambda^{\text{even}} E^*$ from the fundamental forms, without any knowledge of Clifford algebras.

4.9 Hermitian symmetric spaces.

4.9.1 Remark. Note that in the examples of Grassmannians and Spinor varieties above, $|\mathbb{FF}^k| = |\mathbb{FF}^2|^{(k-1)}$ where for a general variety the prolongation property only implies $|\mathbb{FF}^k| \subseteq |\mathbb{FF}^2|^{(k-1)}$. Among homogeneous varieties, there is a preferred subclass, the *Hermitian symmetric spaces*.

4.9.2 Definition. Let $X = G/P \subset \mathbb{P}V$ be a homogeneous variety. Let H be the semi-simple part of P . We say X is a *Hermitian symmetric space* if $T_x X$ is irreducible as an H -module. (This is a nonstandard definition, normally the definition is made using a Riemannian metric.)

4.9.3 Theorem [LM1]. *Let $X = G/P \subset \mathbb{P}V$ be a Hermitian symmetric space in its fundamental embedding and let $x \in X$. Then for $k \geq 2$,*

$$|\mathbb{FF}_{X,x}^{k+1}| = |\mathbb{FF}_{X,x}^2|^{(k-1)}$$

For the proof, see [LM1]. This strict prolongation property fails for a general homogeneous space, e.g., it does not hold for $G_{Q-\text{null}}(k, m)$ for $2 \leq k < [\frac{m}{2}]$. This property has the following geometric consequence (see [LM1] for the proof):

4.9.4 Corollary [LM1]. *Let X be a Hermitian symmetric space in its fundamental embedding, and let $x \in X$. Then*

$$\text{Base}|\mathbb{FF}_{X,x}^k| = \sigma_{k-1}(\text{Base}|\mathbb{FF}_{X,x}^2|).$$

4.10 Fundamental forms of Segres.

Let $U = \mathbb{C}^{a+1}, W = \mathbb{C}^{b+1}$ and consider the Segre, $\text{Seg}(\mathbb{P}U \times \mathbb{P}W) \subset \mathbb{P}(U \otimes W)$. Here we use $G = GL(U) \times GL(W) \subset GL(U \otimes W)$ frames. Let $(e_0, \dots, e_a), (f_0, \dots, f_b), \omega_0^\alpha, \eta_0^j$, $1 \leq \alpha \leq a, 1 \leq j \leq b$ denotes bases and dual bases. We compute

$$\begin{aligned} \underline{d}(e_0 \otimes f_0) &\equiv \omega_0^\alpha e_\alpha \otimes f_0 + \eta_0^j e_0 \otimes f_j \bmod e_0 \otimes f_0 \\ \underline{d}^2(e_0 \otimes f_0) &\equiv \omega_0^\alpha \eta_0^j e_\alpha \otimes f_j \end{aligned}$$

We see II surjects onto N and

$$(4.10.1) \quad |\mathbb{FF}^2| = I_2(\mathbb{P}(e_0 \otimes (W \setminus \{f_0\}))) \sqcup \mathbb{P}((W \setminus \{e_0\}) \otimes f_0))$$

More generally, consider $X = \text{Seg}(\mathbb{P}^{r_1} \times \dots \times \mathbb{P}^{r_d}) = \text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)$, $\dim W_i = r_i$. We use the $GL(W_1) \times \dots \times GL(W_d) \subset GL(W_1 \otimes \dots \otimes W_d)$ frame bundle.

Let $(e_{0j}, \dots, e_{r_j j})$ denote an element of the frame bundle for $\mathbb{P}W_j$ and write the j -th block diagonal element of the Maurer-Cartan form as $\Omega_j = (\omega_{Bj}^{A_j})$

4.10.2 Exercises.

1. Show \mathbb{FF}^2 is the complete system with the base locus the disjoint union of the linear spaces $\mathbb{P}(e_{01} \otimes \dots \otimes e_{0(j-1)} \otimes (W_j/e_{0j}) \otimes e_{0(j+1)} \otimes \dots \otimes e_{0r})$.

2. Show

$$(4.10.2.1)$$

$$\begin{aligned} |\mathbb{FF}^k| = & I_k(\cup_j \mathbb{P}(e_{01} \otimes \dots \otimes e_{0(j_1-1)} \otimes (W_{j_1}/e_{0j_1}) \otimes e_{0(j_1+1)} \otimes \dots \otimes e_{0(j_2-1)} \\ & \otimes (W_{j_2}/\{e_{0j_2}\}) \otimes e_{0(j_2+1)} \otimes \dots \otimes e_{0(j_k-1)} \otimes (W_{j_k}/\{e_{0j_k}\}) \otimes e_{0(j_k+1)} \otimes \dots \otimes e_{0r})) \end{aligned}$$

In particular, the last nonzero fundamental form is \mathbb{FF}^r .

4.10.3 Proposition. *Let $X_j \subset \mathbb{P}W_j$, $1 \leq j \leq r$, be varieties and let $y = [x_1 \otimes \dots \otimes x_r] \in Y := \text{Seg}(X_1 \times \dots \times X_r)$. Then*

$$(4.10.3.1)$$

$$\begin{aligned} \hat{T}_y Y &= \Sigma_j x_1 \otimes \dots \otimes x_{j-1} \otimes \hat{T}_{x_j} X_j \otimes x_{j+1} \otimes \dots \otimes x_r \\ II_{Y,y} &= \Sigma_{j,k} x_1 \otimes \dots \otimes x_{j-1} \otimes \hat{T}_{x_j} X_j \otimes x_{j+1} \dots \otimes x_{k-1} \otimes \hat{T}_{x_k} X_k \otimes x_{k+1} \otimes \dots \otimes x_r \\ &+ \Sigma_j x_1 \otimes \dots \otimes x_{j-1} \otimes \hat{II}_{X_j, x_j} \otimes x_{j+1} \otimes \dots \otimes x_r \end{aligned}$$

4.11 Fundamental forms of the Severi varieties.

Here, it is easier to use the coordinate definition. Choose affine coordinates based at $[p]$ where

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and following the notation of (3.1), denote the affine coordinates $u_1, u_2, u_3 \in \mathbb{A}, r_2, r_3 \in \mathbb{C}$ where the tangent space to p is $\{u_1, u_2\}$ (the span is taken over \mathbb{C}). In these coordinates:

$$\begin{aligned} r_2(u_1, u_2) &= u_1 \bar{u}_1 \text{ as } \det \begin{pmatrix} 1 & \bar{u}_1 \\ u_1 & r_2 \end{pmatrix} = 0 \\ r_3(u_1, u_2) &= u_2 \bar{u}_2 \text{ as } \det \begin{pmatrix} 1 & \bar{u}_2 \\ u_2 & r_3 \end{pmatrix} = 0 \\ u_3(u_1, u_2) &= \bar{u}_2 u_1 \text{ as } \det \begin{pmatrix} 1 & \bar{u}_1 \\ u_2 & u_3 \end{pmatrix} = 0 \end{aligned}$$

where the last equation gives us one, two, four or eight quadratic forms. The determinants come from the vanishing of 2×2 minors that must be zero to make the Severi variety consist only of rank one elements. In division algebra notation the second fundamental forms are

$$(4.11.1) \quad |II| = \mathbb{P}\{u_1 \bar{u}_1, u_2 \bar{u}_2, \bar{u}_2 u_1\}.$$

Note that since this is all of the Taylor series, there are no other differential invariants.

4.11.2 Remark. The alert reader might have observed that, in all cases above, the base locus of the second fundamental form corresponds to the homogeneous space one obtains by marking the nodes in the dynkin diagram of $H = P^{ss}$ adjacent to the nodes removed that correspond to P . This is part of a general pattern, see [LM1] for details.

4.12 Proof of the equality in Terracini's lemma.

Consider the mapping

$$\begin{aligned} \mathcal{F}_Y^1 \times \mathcal{F}_Z^1 &\rightarrow \mathbb{P}V \\ ((e_0, \dots, e_N), (f_0, \dots, f_N)) &\mapsto [e_0 + f_0] \end{aligned}$$

The image is $S(Y, Z)$. We compute its tangent space. Let ω_B^A, η_B^A , denote the entries of the Maurer Cartan forms over Y and Z respectively. Let $1 \leq \alpha \leq \dim Y, 1 \leq j \leq \dim Z$. Note that $\hat{T}_{[e_0]}Y = \{e_0, e_\alpha\}$, $\hat{T}_{[f_0]}Z = \{f_0, f_j\}$. Since we know the line $\mathbb{P}\{e_0, \eta_0\} = \mathbb{P}_{[e_0][f_0]}^1$ is contained in the tangent space to $[e_0 + f_0]$ we calculate modulo $\{e_0, \eta_0\}$.

$$d(e_0 + f_0) \equiv \omega_0^\alpha e_\alpha + \eta_0^j f_j \pmod{\{e_0, f_0\}}$$

Thus $\dim S(X, Z) \leq \dim \{e_0, f_0, e_\alpha, f_j\}$ and by the inequality case they are equal. Moreover $\tilde{T}_{[e_0+f_0]}S(Y, Z) = \mathbb{P}\{\hat{T}_{[e_0]}Y + \hat{T}_{[f_0]}Z\}$. (One can prove the inequality case by observing that the forms $\{\omega_0^\alpha, \eta_0^j\}$ are all independent.)

§5. VARIETIES WITH DEGENERATE GAUSS IMAGES

The new results in this section are from [AGL], currently under preparation.

5.1 Examples.

5.1.1 Joins. Joins have degenerate Gauss maps with at least $(k - 1)$ -dimensional fibers because Terracini's lemma implies that the tangent space to $S(Y_1, \dots, Y_k)$ is constant along each $\mathbb{P}_{y_1, \dots, y_k}^{k-1}$.

Note the special cases of cones, where one of the factors in the join is linear, and of secant varieties, where all factors are the same.

5.1.2 Tangential varieties. Let $X = \tau(Y)$ be a tangential variety. Let $x \in X$ and write $x = [e_0 + e_1]$ with $[e_0] \in Y$ and $[e_1] \in \tilde{T}_y Y$. Then $\tilde{T}_x X = \tilde{T}_{[e_0+te_1]} X$ for all $t \neq 0$.

More generally, if $y \in \tau^k(X) := \bigcup_{x \in X} \tilde{T}_x^{(k)} X$, write $y = [e_0 + e_1 + e_{\mu_1} + \dots + e_{\mu_{k-1}}]$, then $\tilde{T}_y \tau^k(X)$ is constant on the $\mathbb{P}^k \subset \tau^k(X)$ spanned by $\{e_0, e_1, e_{\mu_1}, \dots, e_{\mu_{k-1}}\}$.

5.1.3 Unions of conjugate lines. Let $Y^{n-1} \subset \mathbb{P}^{n+1}$ be a variety with a generic second fundamental form. In this case there exist $n - 1$ simultaneous eigen-directions for the second fundamental form (To make the notion of eigen-direction precise, choose a nondegenerate quadric in II to identify T with T^* and consider the quadrics as endomorphisms of T . The result is independent of the choices.) Let $X^n \subset \mathbb{P}^{n+1}$ be the union of one of these families of embedded tangent lines. Such lines are called *conjugate lines*.

In this case $\gamma(X)$ is degenerate with one dimensional fibers. In fact, the embedded tangent space of X is constant along the conjugate lines. To see this we calculate the tangent space of X using frames over Y . We restrict to frames of Y such that the basis of $T_{[e_0]} Y$ consists of conjugate directions and let $1 \leq i, j \leq n - 1 = \dim Y$. Adapt frames such that $II(e_i, e_j) = \delta_{ij}(e_n + \lambda_i e_{n+1})$. We calculate ($2 \leq \rho \leq n - 1$)

$$d(e_0 + te_1) \equiv (\omega_0^\rho + t\omega_1^\rho)e_\rho + t\omega_0^1(e_n + \lambda_1 e_{n+1}) \bmod \{e_0, e_1\}$$

Thus for generic values of t ,

$$\hat{T}_{[e_0+te_1]} X = \{e_0, e_1, e_\rho, e_n + \lambda_1 e_{n+1}\}.$$

and thus $\hat{T}_{[e_0+te_1]} X$ is independent of t .

Note that if e_1 is not a conjugate direction and one attempts the same construction, the resulting variety will not in general have a degenerate Gauss map.

5.2 Results on Gauss maps.

Let $X^n \subset \mathbb{P}^{n+a} = \mathbb{P}V$ be a variety.

- i. Consider the distribution $\Delta \subset TX$ defined by $\text{singloc}|II|_x$. This distribution is integrable and its integral manifolds are the fibers of the Gauss map γ .
- ii. [??] The generic fibers of γ are linear spaces.
- iii. [??] If $\gamma(X)$ is degenerate, then X is not smooth. More precisely, let F be a general fiber, then X is singular along a codimension one subset in F .
- iv. [Zak] If X is smooth, then γ is finite. (iii. only implies γ is generically finite.)
- v. [Ran] If γ is generically finite, then it is finite.

Although Zak's theorem subsumes iii. I include it for historical reasons and because the methods of proof are quite different and hold the possibility of generalizations in various directions.

5.3 Remark: differences between affine and projective spaces. Ran's arguments are valid in a more general ambient space than a projective space because he does not use the linearity of the fiber, but he does use compactness in an essential way as can be seen by the smooth surface in affine space $z = xy^2 + (1-x)y$, which has the x -axis as an isolated fiber. See [R] for his argument.

5.4 Proofs.

Proof of i. From the definition of II as the derivative of the Gauss map, the first assertion is immediate. A proof is included anyway, as it will serve as a model for more complicated proofs and I do not prove the equivalence of the definitions of II .

Adapt frames to the flag

$$(5.4.1) \quad \hat{x} \subset \{\hat{x}, \text{singloc } |II|_x\} \subset \hat{T} \subset V$$

by letting $\{e_1, \dots, e_f\} \simeq \text{singloc } |II|$. Use additional index ranges $1 \leq s, t \leq f$, $f+1 \leq i, j, k \leq n$. Our adaptations have the effect that $\omega_s^\mu = 0$. The first assertion is equivalent to showing the distribution $\{\omega_0^j\}$ is integrable. Using the Maurer-Cartan equation, we see

$$(5.4.2) \quad d\omega_0^j \equiv -\omega_s^j \wedge \omega_0^s \pmod{\{\omega_0^i\}}$$

We examine the forms ω_s^j . Since $\omega_s^\mu = 0$,

$$(5.4.3) \quad 0 = d\omega_s^\mu = -\omega_j^\mu \wedge \omega_s^j$$

Since we are assuming $\{e_s\}$ is the entire singular locus, for each j , there exists some μ with $\omega_j^\mu \neq 0$ which implies $\omega_s^j = C_{sk}^j \omega_0^k$ for some functions C_{sk}^j . Thus $d\omega_0^j \equiv 0 \pmod{\{\omega_0^i\}}$ and the Frobenius theorem implies the distribution is integrable. \square

Proof of ii. To see the fiber is a linear space, it will suffice to show that $II_{F,[e_0]} = 0$. On F , $\omega_0^j = 0$ so we have

$$(5.4.4) \quad \begin{aligned} de_0 &\equiv \omega_0^s e_s \pmod{e_0} \\ \underline{d}^2 e_0 &\equiv \omega_s^j \omega_0^s e_j \pmod{\{e_0, e_s\}} \\ &\equiv 0 \pmod{\{e_0, e_s\}} \text{ and } \pmod{\{\omega_0^j\}} \quad \square \end{aligned}$$

Proof of iii. Fix a fiber F of γ and let $p = u^0 e_0 + u^s e_s \in F$. We calculate $\tilde{T}_p X$. Since $F \subset \tilde{T}_p X$ for all $p \in F$ we can work modulo $F = \mathbb{P}\{e_0, e_s\}$.

$$(5.4.5) \quad \begin{aligned} dp &\equiv (u^0 \omega_0^j + u^s \omega_s^j) e_j \pmod{\{e_0, e_s\}} \\ &\equiv (u^0 \delta_j^k + u^s C_{sj}^k) \omega_0^j e_k \end{aligned}$$

We may think of $[u^0, \dots, u^f]$ as parametrizing a $(f-1)$ -dimensional (projective dimension) family of matrices, which must drop rank at least along a codimension one subset, which corresponds to $(X \cap F)_{\text{sing}} \subseteq X_{\text{sing}}$. \square

An important feature of this proof is that we were able to calculate the tangent space to F using first and second order information at just one point of F , namely $[e_0]$. We will see this type of calculation reappear when studying $\tau(X)$ and X^* .

Second proof of iii. Let F denote a typical fiber of the Gauss map of X . We have

$$(5.4.6) \quad \begin{array}{ccc} F & \longrightarrow & X \\ & & \downarrow \\ & & \gamma(X) \end{array}$$

which determines a map

$$(5.4.7) \quad \begin{aligned} \gamma(X) &\rightarrow \mathbb{G}(k, \mathbb{P}V) \\ \gamma(x) &\mapsto F_{\gamma(x)} = \gamma^{-1}\gamma(x) \end{aligned}$$

The differential of this mapping is a linear map $T_{\gamma(x)}\gamma(X) \rightarrow \hat{F}_{\gamma(x)}^* \otimes (\hat{T}_x X / \hat{F}_{\gamma(x)})$ which can be thought of as a map $\hat{F}_{\gamma(x)} \rightarrow T_{\gamma(x)}\gamma(X)^* \otimes T_x X / \hat{F}_{\gamma(x)}$, from a k -dimensional vector space to the space of $(n-k) \times (n-k)$ matrices. If $T_x X$ is to have the proper dimension, the image matrix must be of full rank for all $f \in F_{\gamma(x)}$, but this is impossible, one cannot have a linear space of square matrices such that all are of maximal rank, in fact the matrices must drop rank along a codimension one subset. \square

5.5 Exercise. Prove that a smooth variety X cannot be ruled by \mathbb{P}^k 's for $k \geq a$ by considering the reduced Gauss map $B^{n-k} \rightarrow \mathbb{G}(k, \mathbb{P}V)$ where B is the base space. A generalization of this result is given in §6.

§6. SMOOTHNESS OF RULED AND UNIRULED VARIETIES

In this section I discuss codimension restrictions on ruled and uniruled varieties. Ruled varieties are special cases of varieties that can be described as fibrations, and the bound on their codimension is the same for that of an arbitrary fibered variety:

6.1 Theorem, Remmert and Van de Van [RV]. *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety that is a fibration with fibers of dimension f , then $f < a$.*

The proof of Remmert and Van de Van's theorem is a simple application of the intersection property of projective space. Let $p \in B$ be a point and let $Z \subset B$ be a hypersurface not containing p . $\dim \pi^{-1}(p) = f$ and $\dim \pi^{-1}(Z) = (n-f-1) + f = n-1$. Thus if $f+n-1 > n+a$, i.e. $f \geq a$, $\pi^{-1}(p) \cap \pi^{-1}(Z) \neq \emptyset$ which is a contradiction.

If X^n is only uniruled by k -planes, it is easier to be smooth, for example the smooth quadric hypersurface Q^n is uniruled by $[\frac{n}{2}]$ -planes. Nevertheless there are still restrictions on the codimension:

6.2 Theorem [Ran, R3]. *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety such that through a general point $x \in X$ there is at least one $\mathbb{P}^k \subset X$ containing x . Then $a \geq k/(n-k)$.*

Ran's proof follows from his discussion of generalized Gauss maps. He also remarks that the results can be derived from the Barth-Larsen theorems. Here is an alternative proof that is local in character, in that the only way global information is taken into account is that the singular locus of the second fundamental form must be empty:

Proof. We are given that at each point there is a \mathbb{P}^{k-1} contained in $\text{Base}[II]$. The proof will follow from the following lemma:

6.3 Lemma. *Let A be an a -dimensional system of quadrics on an n -dimensional vector space V such that there is a linear space W of dimension k in the base locus of A . If $a < k/(n-k)$, then A has a singular locus.*

Proof. Any quadric $q \in A$ can be written

$$q = v^1 w^1 + \dots + v^k w^k + q'$$

where, $q' \in S^2 W^\perp$, and $v^j \in W^\perp$. Since $k > n-k$, at most $n-k$ of the v^j are independent. Thus each quadric has at least an $k - (n-k) = 2k - n$ dimensional singular locus in W , so if $a(n-k) < k$ then $\text{singloc}(A) \neq 0$. \square

6.4 Problem. Use the rank restriction theorem to get a better bound. (Although among bounds of the form above, Ran's bound is optimal as equality cases occur.)

§7. VARIETIES WITH DEGENERATE DUAL VARIETIES

Let $X^n \subset \mathbb{P}^{n+a} = \mathbb{P}V$ be a variety, let $X^* \subset \mathbb{P}V^*$ denote its dual variety, and let $\delta_* = \delta_*(X) = n+a-1-\dim X^*$ denote the dual defect, as discussed in the introduction.

7.1 Examples.

- i. The smooth quadric hypersurface is self dual.
- iii. Let $X = \text{Seg}(\mathbb{P}W_1 \times \mathbb{P}W_2) = \text{Seg}(\mathbb{P}^k \times \mathbb{P}^l)$, with $k \geq l$, then $X^* = \sigma_l(\text{Seg}(\mathbb{P}W_1^* \times \mathbb{P}W_2^*))$ and thus X^* is degenerate iff $k \neq l$ with defect $\delta_* = k-l$
- iii'. More generally, if $X = \text{Seg}(\mathbb{P}^{k_1} \times \dots \times \mathbb{P}^{k_r})$ where $k_1 \geq k_2 \geq \dots \geq k_r$, then X^* is degenerate iff $k_1 > k_2 + \dots + k_r$ with defect $\delta_* = k_1 - (k_2 + \dots + k_r)$.
- iv. If X is a scroll, that is a linear fibration with base a curve, then $\delta_* = n-2$.
- vi. If $X = G(2, W) \subset \mathbb{P}\Lambda^2 W$, then $X^* = \sigma_p(G(2, W^*))$, where $p = \frac{1}{2}(k-2)$ if k is even and $\frac{1}{2}(k-3)$ if k is odd, thus $\delta_* = 0$ if k is even and 2 if k is odd.
- vii. $\mathbb{S}_5(V, Q)^* = \mathbb{S}_5(V^*, Q)$, $\delta_* = 4$. (Higher Spinor varieties have nondegenerate duals). Note that the example of the Segre shows that there can be no absolute bound on δ_* .

7.2 Theorems on dual varieties. *Let $X^n \subset \mathbb{P}^{n+a} = \mathbb{P}V$ be a variety and let $X^* \subset \mathbb{P}V^*$ denote its dual variety. Let $\delta_* = n+a-1-\dim X^*$ denote the dual defect of X .*

- i. [Bertini] $(X^*)^* = X$ (the reflexivity theorem)
- ii. $\delta_* \geq \delta_\gamma := \dim X - \dim \gamma(X)$.

Assume X is not contained in a hyperplane.

- iii.
 - a. If X is a smooth hypersurface, X^* is nondegenerate.
 - b. If X is any curve, then X^* is nondegenerate.
 - c. If X is any surface, then X^* is nondegenerate.
- iv. If X is ruled by k -planes, i.e. if X is a linear fibration, with k -dimensional fibers, then $\delta_* \geq 2k - n$.
 - v. If X is a cone over a linear space L , then $X^* \subset L^\perp$.
 - vi. [Bertini] If $H \in X_{sm}^*$, then $\{x \in X \mid \tilde{T}_x X \subset H\}$ is a linearly embedded $\mathbb{P}^\delta \subset \mathbb{P}V$.
 - vii. [Zak, Z] If X is smooth, then $\dim X^* \geq \dim X$, i.e. $\delta_* \leq a - 1$.
 - viii. [Landman, E1] If X is smooth, then $n - \delta_*$ is even. In particular $\delta_* = n - 1$ is impossible.
 - ix. [Ein, E1] If X is smooth, $\dim X = \dim X^*$, $X \not\subset \mathbb{P}^{n+a-1}$, and $a \geq \frac{n}{2}$, then X is one of the following:
 - a. $\text{Seg}(\mathbb{P}^1 \times \mathbb{P}^{n-1}) \subset \mathbb{P}^{2n-1}$.
 - b. $G(2, 5) \subset \mathbb{P}^9$.
 - c. $\mathbb{S}_5^{10} \subset \mathbb{P}^{15}$.

Moreover, in these cases X^* is isomorphic to X .

- x. [Ein, E1] [BC]. If $a = 2$, $X \not\subset \mathbb{P}^{n+a-1}$, and X is smooth, then X^* is a hypersurface unless $X = \text{Seg}(\mathbb{P}^1 \times \mathbb{P}^2) \subset \mathbb{P}^5$.
- xi. [Ein, E2]. If $\delta_* = n - 2$ then X is a scroll.
- xii. [Ein, E2]. If $\delta_* \geq \frac{n}{2}$ then X is ruled by $\mathbb{P}^{\frac{\delta_*+n}{2}}$'s.
- xiii. [Ein, E2, 1.3]. If $X = Y \cap Z$ with Y a smooth hypersurface of degree greater than one, then X^* is nondegenerate.
- xiv. If H is a hyperplane, then $\delta_*(X \cap H) = \delta_*(X) - 1$.

Aluffi has proved a partial converse to vi, see [Aluffi].

There are also classification results in small dimension see [E2] and [BFLS].

Landman's original (unpublished) proof of viii. used Lefshetz pencils.

Ein's proofs rely on studying the conormal bundle of $Y_H := \{x \in X \mid \tilde{T}_x X \subset H\}$ and its deformations.

I do not know of a direct geometric proof of vi, both Ein and Ballico and Chianti use facts about rank two vector bundles on projective space.

7.3 Theorems on the projective differential geometry of X^* . Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and let $x \in X$ be a general point. Let $\delta_* = n + a - 1 - \dim X^*$ denote the dual defect of X . Given $H \in X^*$, let $Y_H = \{x \in X \mid \tilde{T}_x X \subset H\}$.

- i. (??) [GH], [L3] $|II_{X,x}|$ is a system of quadrics of projective dimension $a - 1$ and bounded rank $n - \delta_*$ on an n -dimensional vector space. I.e. If r is the rank of a generic quadric in $|II_{X,x}|$, then $\delta_* = n - r$.
- ii. [IL, 3.4] If X is smooth and $H \in X_{sm}^*$, then $|II_{X^*,H}|$ is a system of quadrics of projective dimension δ_* and constant rank $n - \delta_*$ on an $(n - \delta_* + a - 1)$ -dimensional vector space. I.e. If r is the rank of the quadrics in $|II_{X^*,H}|$, then $\delta_* = n - r$.
- iii. [IL] If $H \in X^*$ is a smooth point, then $|II|_{X^*,H}$ can be recovered from $II_{X,x}$ as x ranges over Y_H . (The precise inversion formula is given below.)
- iv. [IL] If $H \in X^*$ is a smooth point, then $|II|_{X^*,H}$ can be recovered from $II_{X,x}$ and $F_{3X,x}$ where $x \in Y_H$ is any smooth point. (The precise inversion formula is given below.)

v. [L7] Let $X^n \subset \mathbb{P}^{n+a}$ be an analytic open subset of a variety. If $a \geq 2$, X is not contained in a hyperplane, and for general $x \in X$ there exists a linear space L_x^{n-1} osculating to order two at x , then $L_x \subset \overline{X}$. In particular, any smooth open subset of a variety having the second fundamental form of a scroll is a scroll.

vi. [L7] Let $X^n \subset \mathbb{P}^{n+a}$ be an analytic open subset of a variety having the property that through a general point $x \in X$, there exists a linear space L_x , of dimension k , osculating to order two at x . Then there exists a linear subspace $M_x \subset L_x$, of dimension $2k - n$, such that $M_x \subset \overline{X}$. In particular $\delta_* \geq 2k - n$.

vii. [IL] If there exists $x \in X_{sm}$ such that $Y_x \cap X^* \subset X_{sm}^*$. Then $\delta_* \geq a - 1$.

viii. [IL] If there exists $x \in X_{sm}$ such that $Y_x \cap X^* \subset X_{sm}^*$ and $H \in X_{sm}^*$ such that $Y_H \cap X \subset X_{sm}$, Then $\dim X = \dim X^*$.

iv may be considered as an infinitesimal version of 7.2.vii. 7.3.v is related to, but different from 7.2.viii.

7.4 Frames for X and X^* .

Let $H \in X_{sm}^*$ and let

$$Y_H = \{x \in X \mid \tilde{T}_x X \subseteq H\}$$

Let

$$\begin{aligned} \mathcal{I}^0 &= \{(x, H) \mid x \in X_{sm}, H \in X_{sm}^*, \tilde{T}_x X \subseteq H\} \\ &= \{(x, H) \mid x \in X_{sm}, H \in X_{sm}^*, \tilde{T}_H X^* \subseteq x\} \\ &= \{(x, H) \mid x \in X_{sm}, H \in X_{sm}^*, x \in Y_H\} \end{aligned}$$

and let $\mathcal{F}^* \rightarrow \mathcal{I}^0$ be the frame bundle of bases of V over $(x, H) \in \mathcal{I}^0$ adapted to the flag:

$$(7.4.1) \quad 0 \subset \hat{x} \subset \hat{T}_x Y_H \subset \hat{T}_x X \subset \hat{H}^\perp \subset V.$$

To describe \mathcal{F}^* using indices, let $1 \leq \alpha, \beta \leq n$, $1 \leq s, t, u \leq n-r$, $n-r+1 \leq i, j, k \leq n$, $1 \leq \kappa, \lambda \leq a-1$. Require that

$$\begin{aligned} \hat{x} &= \{e_0\} \\ \hat{T}_x Y_H &= \{e_0, e_s\} \\ \hat{T}_x X &= \{e_0, e_s, e_j\} = \{e_0, e_\alpha\} \\ \hat{H} &= \{e_0, e_\alpha, e_{n+\lambda}\} \end{aligned}$$

In indices, the flag (7.4.1) is

$$(7.4.2) \quad \{e_0\} \subset \{e_0, e_s\} \subset \{e_0, e_s, e_j\} = \{e_0, e_\alpha\} \subset \{e_0, e_\alpha, e_{n+\lambda}\} \subset \{e_0, e_\alpha, e_{n+\lambda}, e_{n+a}\} = V$$

Write the pullback of the Maurer-Cartan form to \mathcal{F}^* as:

$$(7.4.3) \quad \Omega = \begin{pmatrix} \omega_0^0 & \omega_t^0 & \omega_k^0 & \omega_{n+\lambda}^0 & \omega_{n+a}^0 \\ \omega_0^s & \omega_k^s & \omega_t^s & \omega_{n+\lambda}^s & \omega_{n+a}^s \\ \omega_0^j & \omega_k^j & \omega_t^j & \omega_{n+\lambda}^j & \omega_{n+a}^j \\ 0 & \omega_t^{n+\kappa} & \omega_k^{n+\kappa} & \omega_{n+\lambda}^{n+\kappa} & \omega_{n+a}^{n+\kappa} \\ 0 & 0 & \omega_k^{n+a} & \omega_{n+\lambda}^{n+a} & \omega_{n+a}^{n+a} \end{pmatrix}$$

where $\omega_t^{n+a} = 0$ because we have adapted such that $\{\underline{e}_t\} = \text{singloc}(q^{n+a})$, which proves (7.3.i).

We first prove Y_H is a linear space, proceeding as we did in §5 to show that the fibers of the Gauss map are linear spaces. (Alternatively one can argue that if $H \in X_{sm}^*$, then $\rho^{-1}(H) = \mathbb{P}N_H^*X^*(1) = \mathbb{P}^\delta$ and then that π linearly embeds the \mathbb{P}^δ .) First note that

$$(7.4.4) \quad 0 = d\omega_s^{n+a} = -\omega_k^{n+a} \wedge \omega_s^k$$

The forms $\omega_k^{n+a} = q_{ki}^{n+a}\omega_0^i$ are all independent since q^{n+a} descends to be a nondegenerate quadratic form on $T_x X / T_x Y_H$, and $\hat{T}_x Y_H \simeq N_H^*X^*(1)$ is of constant dimension. Thus $\omega_s^k \equiv 0 \pmod{\{\omega_0^j\}}$.

We calculate $II_{Y_H, [e_0]}$:

$$(7.4.5) \quad \begin{aligned} de_0 &\equiv \omega_0^s \pmod{e_0 \text{ and } \{\omega_0^j\}} \\ \underline{d}^2 e_0 &\equiv \omega_s^j \omega_0^s \pmod{\{e_0, e_s\} \text{ and } \{\omega_0^j\}} \\ &\equiv 0 \pmod{\{e_0, e_s\} \text{ and } \{\omega_0^j\}} \end{aligned}$$

Pulled back to Y_H , $\omega_0^j = 0$, which implies $II_{Y_H, [e_0]} = 0$ and thus Y_H is a linear space.

We may also consider \mathcal{F}^* as an adapted frame bundle over X_{sm}^* . It is clearly zero-th order adapted, but is in fact as adapted to X^* as it is to X . To see this, note that \mathcal{F}^* is also adapted to the dual flag:

$$(7.4.6) \quad 0 \subset \hat{H} \subset (\hat{T}_x X)^\perp \subset (\hat{T}_x Y_H)^\perp \subset \hat{x}^\perp \subset V^*.$$

But this flag is just

$$(7.4.7) \quad 0 \subset \hat{H} \subset \hat{Y}_x \subset \hat{T}_H X^* \subset \hat{x}^\perp \subset V^*.$$

To consider \mathcal{F}^* as an adapted frame bundle over X_{sm}^* , let $\{e^B\}$ denote the dual basis to $\{e_B\}$. Let $\langle \cdot, \cdot \rangle$ denote the pairing $V \times V^* \rightarrow \mathbb{C}$, so $\langle e_A, e^B \rangle = \delta_A^B$. We calculate:

$$(7.4.8) \quad 0 = d \langle e_A, e^B \rangle = \langle \omega_C^A e_A, e^B \rangle + \langle e_A, de^B \rangle$$

which implies

$$(7.4.9) \quad de^B = -\omega_C^B e^C.$$

We may specify a point of \mathcal{F}^* using the dual basis e^B and it will be convenient to do so when studying frames over X^* . From this dual perspective, $H = [e^{n+a}]$, $Y_x = \mathbb{P}\{e^{n+a}, e^{n+\lambda}\}$, $\tilde{T}_H X^* = \mathbb{P}\{e^{n+a}, e^{n+\lambda}, e^k\}$, $x = \mathbb{P}\{e^{n+a}, e^{n+\lambda}, e^\alpha\}$.

To see $\tilde{T}_H X^* = \mathbb{P}\{e^{n+a}, e^k, e^{n+\lambda}\}$, note that

$$(7.4.10) \quad de^{n+a} \equiv -\omega_k^{n+a} e^k - \omega_{n+\lambda}^{n+a} e^{n+\lambda} \pmod{e^{n+a}}.$$

(7.4.10) shows that $\{\omega_k^{n+a}, \omega_{n+\lambda}^{n+a}\}$ form a basis of the semi-basic forms for the projection to X^* , note that $\dim\{\omega_k^{n+a}\} = r$. It also shows that on \mathcal{F}^* , the forms $\omega_{n+\lambda}^{n+a}$ are independent and independent of the forms ω_0^j because the dimension of the tangent space to X^* is constant at all smooth points of X^* . If X is smooth, then (7.4.10) is valid for any $H \in X_{sm}^*$.

Using (4.4), we compute $II_{X^*,H}$:

(7.4.11)

$$\begin{aligned} \underline{d}^2 e^{n+a} &\equiv (\omega_k^{n+a} \omega_s^k + \omega_{n+\lambda}^{n+a} \omega_s^{n+\lambda}) e^s + \omega_k^{n+a} \omega_0^k e^0 \bmod \hat{T}_H X^* \\ &\equiv (r_{sij}^{n+a} \omega_0^j + 2q_{si}^{n+\lambda} \omega_{n+\lambda}^{n+a}) \omega_0^i e^s + q_{jk}^{n+a} \omega_0^j \omega_0^k e^0 \bmod \hat{T}_H X^* \end{aligned}$$

where to derive the second line we utilized:

$$(7.4.12) \quad r_{st\beta}^{n+a} \omega_0^\beta = 0$$

$$(7.4.13) \quad r_{sij}^{n+a} \omega_0^j = -q_{si}^{n+\lambda} \omega_{n+\lambda}^{n+a} + q_{kj}^{n+a} \omega_s^k.$$

The second line of (7.4.11) proves (7.3.iii), (7.3.iv) and (7.3.ii); the first because Q_x is the quadric corresponding to e^0 and the second because all the other invariants appearing are coefficients either of $II_{X,x}$ or $F_{3X,x}$. The first inversion formula in turn proves [IL, 3.4] because the rank of the quadric corresponding to e^0 is of rank r , but as one moves in $N_H^* X^*$, every conormal direction gets a chance to be e^0 (under the hypothesis $H \cap X_{sing} = \emptyset$).

Note that (7.4.10), (7.4.11) imply that the dual variety of a smooth variety has the property that an arbitrary smooth point of X^* is “general” to order two.

By a classical theorem (see the next section), it is not possible to have a system of quadrics of constant rank if r is odd, which, combined with (7.3.ii) provides a new proof of the Landman parity theorem (7.2.viii).

[IL, 3.4] led us to study systems of quadrics of bounded and constant rank. Our main result on systems of quadrics is the resolution of the constant even rank problem:

7.4.14 Theorem [IL, 2.16]. *If r is even then*

$$\max \{ \dim(A) \mid A \subset S^2 \mathbb{C}^m \text{ is of constant rank } r \} = m - r + 1.$$

[IL, 2.16] immediately provides a new proof of Zak’s theorem that $\delta_* \leq a - 1$, without using the Fulton-Hansen connectedness theorem but still using the Lefschetz theorem as will be explained in §8.

The discussion above proves the rank restriction theorem (0.3.3.4) for the case of a generic quadric, and the case for an arbitrary quadric follows from counting how dimensions drop, see [L3].

We may state (7.3.iv) in the more precise form:

7.4.15 Inversion formula, [IL]. *With respect to the basis $\{\omega_{n+\lambda}^{n+a}, \omega_0^j\}$ of $T_H^* X^*$, and e^0, e^s of $N_H X^*$, we have*

$$II_{X^*,H} = \left\{ Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & q_{jk}^{n+a} \end{pmatrix}, \quad Q_s = \begin{pmatrix} 0 & q_{sj}^{n+\lambda} \\ q_{sk}^{n+\lambda} & r_{sjk}^{n+a} \end{pmatrix} \right\}$$

where the blocking is $(a-1, r) \times (a-1, r)$.

Regarding (7.4.10), we made the following observation:

7.4.16 Observation. If $\frac{r}{2} < a - 1$, i.e. if $\delta_* < n - 2(a - 1)$, then $II_{X^*, H}$ cannot be of constant rank r unless the $\delta + 1$ matrices $(q_{jk}^{n+a}), (r_{1jk}^{n+a}), \dots, (r_{\delta jk}^{n+a})$ are all linearly independent.

Were the matrices dependent, there would be a quadric in $II_{X^*, H}$ whose lower right $r \times r$ block was zero, but if $\frac{r}{2} < a - 1$, such a quadric could not have rank r .

Observation (7.4.16) motivated the following conjecture:

7.4.17 Conjecture. If $X^n \subset \mathbb{P}^{n+a}$ is a smooth variety with $\delta_* > 0$, then $\delta_* \geq n - 2(a - 1)$.

7.4.18 Corollary of Conjecture. If $X^n \subset \mathbb{P}^{n+a}$ is a smooth variety with $a - 1 < \frac{n}{3}$, then $\delta_* = 0$.

The motivation for the conjecture is that if $\delta_* < n - 2(a - 1)$, there are *genericity* conditions placed on the coefficients of F_3 that appear to be incompatible with the closed conditions arising from the degeneracy of X^* .

§8. SYSTEMS OF BOUNDED AND CONSTANT RANK

8.1 Examples in coordinates. A refers to symmetric systems, B to linear systems, and C to skew symmetric systems. In all cases $[e_0, \dots, e_\delta] \in \mathbb{P}^\delta$. Consider

$$B_I. \begin{pmatrix} e_0 & 0 \\ e_1 & e_0 \\ e_2 & e_1 \\ 0 & e_2 \end{pmatrix} \quad C_{II}. \begin{pmatrix} 0 & e_0 & e_1 \\ -e_0 & 0 & e_2 \\ -e_1 & -e_2 & 0 \end{pmatrix}$$

$$A_I. \begin{pmatrix} 0 & 0 & 0 & 0 & e_0 & 0 \\ 0 & 0 & 0 & 0 & e_1 & e_0 \\ 0 & 0 & 0 & 0 & e_2 & e_1 \\ 0 & 0 & 0 & 0 & 0 & e_2 \\ e_0 & e_1 & e_2 & 0 & 0 & 0 \\ 0 & e_0 & e_1 & e_2 & 0 & 0 \end{pmatrix} \quad A_{II}. \begin{pmatrix} 0 & 0 & 0 & 0 & e_0 & e_1 \\ 0 & 0 & 0 & -e_0 & 0 & e_2 \\ 0 & 0 & 0 & -e_1 & -e_2 & 0 \\ 0 & -e_0 & -e_1 & 0 & 0 & 0 \\ e_0 & 0 & -e_2 & 0 & 0 & 0 \\ e_1 & e_2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{III}. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_0 & e_1 & e_2 \\ 0 & 0 & 0 & 0 & 0 & -e_0 & -e_1 & 0 & 0 & e_3 \\ 0 & 0 & 0 & 0 & -e_0 & 0 & -e_2 & 0 & -e_3 & 0 \\ 0 & 0 & 0 & 0 & -e_1 & -e_2 & 0 & -e_3 & 0 & 0 \\ 0 & 0 & -e_0 & -e_1 & 0 & 0 & 0 & 0 & 0 & e_4 \\ 0 & -e_0 & 0 & -e_2 & 0 & 0 & 0 & 0 & -e_4 & 0 \\ 0 & -e_1 & -e_2 & 0 & 0 & 0 & 0 & -e_4 & 0 & 0 \\ e_0 & 0 & 0 & -e_3 & 0 & 0 & -e_4 & 0 & 0 & 0 \\ e_1 & 0 & -e_3 & 0 & 0 & -e_4 & 0 & 0 & 0 & 0 \\ e_2 & e_3 & 0 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_{IV} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_0 & e_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_0 & e_1 & 0 & e_2 \\ 0 & 0 & 0 & 0 & 0 & -e_0 & e_1 & 0 & e_2 & e_3 \\ 0 & 0 & 0 & 0 & e_0 & e_1 & 0 & e_2 & e_3 & 0 \\ 0 & 0 & 0 & -e_0 & 0 & 0 & e_2 & -e_3 & 0 & 0 \\ 0 & 0 & e_0 & -e_1 & 0 & 0 & e_3 & 0 & 0 & 0 \\ 0 & -e_0 & -e_1 & 0 & -e_2 & -e_3 & 0 & 0 & 0 & 0 \\ -e_0 & -e_1 & 0 & -e_2 & e_3 & 0 & 0 & 0 & 0 & 0 \\ -e_1 & 0 & -e_2 & -e_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -e_2 & -e_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} A_{IV} \cdot \begin{pmatrix} 0 & C_{IV} \\ {}^t C_{IV} & 0 \end{pmatrix}$$

8.2 Examples via standard constructions.

8.2.1 Doubled systems. Given a linear subspace $B \subset \mathbb{C}^k \otimes \mathbb{C}^l$ of constant rank p , one can form a system of symmetric or skew, maps of rank $2p$ in $S^2 \mathbb{C}^{k+l}$ (or $\Lambda^2 \mathbb{C}^{k+l}$), namely

$$\begin{pmatrix} 0 & B \\ {}^t B & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & B \\ -{}^t B & 0 \end{pmatrix}.$$

We call such systems, *doubled* systems. Examples A_I, A_{II}, A_{IV} are doubled systems.

8.2.2 Systems of split type. A suitably generic subspace $B \subset \mathbb{C}^l \otimes \mathbb{C}^k$ of dimension at most $l - k - 1$ ($k \leq l$) is a system of linear maps of constant rank $k - 1$. We will refer to the doublings of such systems as *systems of split type*. The name will be explained below.

8.2.3 Systems of graded algebra type. Consider a vector space V and the inclusion

$$(8.2.3.1) \quad \begin{aligned} V &\rightarrow \text{Hom}(\Lambda^k V, \Lambda^{k+1} V) \\ v &\mapsto (E \mapsto v \wedge E) \end{aligned}$$

We will call these systems, *systems of graded algebra type*. One can also use Clifford multiplication in the construction.

One can double these systems to get symmetric or skew systems. However, it is not always necessary to double: If $\dim V = 2k + 1$, then $\dim \Lambda^k V = \dim \Lambda^{k+1} V$ and we may identify them using a volume form. If k is even, the maps are symmetric, and they are skew symmetric if k is odd (see [IL]). Examples B_{II} and A_{III} arise in this way.

We can also consider systems of bounded rank via

$$\Lambda^p V \rightarrow \text{Hom}(\Lambda^k V, \Lambda^{k+p} V)$$

and one can take subsystems of these to obtain systems of constant rank.

8.2.4 A mystery. Example C_{IV} is due to Westwick [W2]. I do not have a geometric explanation for it, although it might be related to the Horrocks-Mumford bundle.

8.2.5 Examples arising from second fundamental forms of dual varieties. The varieties $\text{Seg}(\mathbb{P}^1 \times \mathbb{P}^n), G(2, 5), \mathbb{S}_5$ are self-dual so their second fundamental forms are of constant rank:

$$\begin{aligned} |II|_{\text{Seg}(\mathbb{P}^1 \times \mathbb{P}^n)^*, x \otimes y} &= I_2(\mathbb{P}(\mathbb{C} \otimes y) \sqcup \mathbb{P}(x \otimes \mathbb{C}^n)) = I_2(\mathbb{P}^0 \sqcup \mathbb{P}^{n-1}) \\ |II|_{G(2,5), E} &= I_2(\text{Seg}(E^* \otimes V/E)) = I_2(\text{Seg}(\mathbb{P}^1 \times \mathbb{P}^2)) \\ |II|_{\mathbb{S}_5, E} &= I_2(G(2, E)) = I_2(G(2, 5)) \end{aligned}$$

From the pattern, one might guess that $I_2(\mathbb{S}_5)$ might be of constant rank. This is not the case, although it is nearly of constant rank. We have already seen that it occurs as the system associated to the second fundamental form of $\mathbb{OP}^2 \subset \mathbb{PH}$ and that only two ranks are possible for the quadrics.

I discuss the problems of arbitrary linear systems and skew-symmetric systems of constant rank as well as symmetric systems of constant rank because the three are related.

8.3 Theorems on systems of constant and bounded rank. Let

$$\begin{aligned} l(r, m, n) &= \max \{ \dim(A) \mid A \subset \mathbb{C}^n \otimes \mathbb{C}^m \text{ is of constant rank } r \} \\ \underline{l}(r, m, n) &= \max \{ \dim(A) \mid A \subset \mathbb{C}^n \otimes \mathbb{C}^m \text{ is of bounded rank } r \} \\ \bar{l}(r, m, n) &= \max \{ \dim(A) \mid A \subset \mathbb{C}^n \otimes \mathbb{C}^m \text{ is of rank bounded below by } r \} \end{aligned}$$

and similarly let $c(r, m)$ (resp. $\lambda(r, m)$) etc... denote the corresponding numbers for symmetric (resp. skew symmetric) matrices.

i. [W] Suppose $2 \leq r \leq m \leq n$. Then

- (1) $l(r, m, n) \leq m + n - 2r + 1$
- (2) $l(r, m, n) = n - r + 1$ if $n - r + 1$ does not divide $(m - 1)!/(r - 1)!$
- (3) $l(r, r + 1, 2r - 1) = r + 1$

ii. (Classical, see [IL, 2.8])

- (1) $\bar{l}(r, m, n) = (m - r)(n - r)$.
- (2) $\bar{c}(r, m) = \binom{m-r+1}{2}$.
- (3) $\bar{\lambda}(r, m) = \binom{m-r}{2}$ (r even).

iii. (Classical, see [IL, 2.10])

- (1) If $0 < r \leq m \leq n$ then $l(r, m, n) \geq n - r + 1$.
- (2) If r is even then $c(r, m) \geq m - r + 1$.
- (3) If r is even then $\lambda(r, m) \geq m - r + 1$.

iv. (Classical, see [IL, 2.15]) If r is odd then $c(r, m) = 1$.

v. [IL, 2.16] If r is even then $c(r, m) = m - r + 1$.

To prove iii, take $A^{n-r+1} \subset \mathbb{C}^n \otimes \mathbb{C}^m$ generic to show iii.1, and double symmetrically and skew-symmetrically to get iii.2 and iii.3.

I recently solved a conjecture of C. Pauly, characterizing the maximal linear spaces of skew symmetric matrices of bounded rank:

8.3.1 Proposition. *If $L \subset \Lambda^2 \mathbb{C}^m$ is of bounded rank $m - 1$ and dimension $\binom{m-1}{2}$, then $L = \Lambda^2 \mathbb{C}^{m-1}$.*

If the dimension of L is not maximal, then the conclusion does not necessarily hold.

8.4 Direct geometric methods. There is a natural geometric way to study systems of bounded and constant rank. Consider $\sigma_r(\text{Seg}(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}))$, $\sigma_r(v_2(\mathbb{P}^{m-1}))$, and $\sigma_{\frac{r}{2}} G(2, m)$. The problem of finding systems of bounded rank is to find linear spaces on these varieties, the problem of finding constant rank systems is to find linear spaces contained in the smooth locus of these varieties, and the problem of finding systems with rank bounded below is to find linear spaces on the ambient projective space avoiding these varieties. To prove ii, one calculates the dimensions of the varieties and use Bezout's theorem.

8.4.1 Problem. Explicitly identify the examples above as linear spaces on $\sigma_r(v_2(\mathbb{P}T)) \setminus \sigma_{r-1}(v_2(\mathbb{P}T))$, $\sigma_{\frac{r}{2}}(G(2, T)) \setminus \sigma_{\frac{r}{2}-1}(G(2, T))$, $\sigma_r(\text{Seg}(\mathbb{P}^m \times \mathbb{P}^n)) \setminus \sigma_{r-1}(\text{Seg}(\mathbb{P}^m \times \mathbb{P}^n))$.

For example, the doubling of a system $B \subset U \otimes W$ with $\dim U = k$, $\dim W = m - k$ to $V = U \oplus W$, must be contained in

$$T_{[x_1^2 + \dots + x_k^2]} \sigma_k(v_2(\mathbb{P}V)) \cap T_{[x_{k+1}^2 + \dots + x_m^2]} \sigma_{m-k}(v_2(\mathbb{P}V))$$

where x_1, \dots, x_m is a basis of V such that $U = \{x_1, \dots, x_k\}$, $W = \{x_{k+1}, \dots, x_m\}$.

8.5 Vector bundle methods.

Let $A \subseteq V^* \otimes W$ be a linear subspace. Over the projective space $\mathbb{P}A$ are the vector bundles $V \otimes \mathcal{O}_{\mathbb{P}A}$, $W \otimes \mathcal{O}_{\mathbb{P}A}(1)$ and a vector bundle map ϕ between them,

$$(8.5.1) \quad \begin{aligned} \phi_{[a]} : V \otimes \mathcal{O}_{\mathbb{P}A} &\rightarrow W \otimes \mathcal{O}_{\mathbb{P}A}(1) \\ v &\mapsto a(v) \otimes a^*. \end{aligned}$$

The image and kernel of ϕ are in general sheaves, but if A is a system of constant rank, they are vector bundles. Let $E := \phi(\mathcal{O}_{\mathbb{P}A} \otimes V)$ denote the image vector bundle in this case. If $V^* \simeq W$ and the system is symmetric or skew, then one deduces that $E \simeq E^*(1)$ by considering the dual sequence.

By Grothendieck's theorem (a proof of which is already in [HP]), any vector bundle $E \rightarrow \mathbb{P}^1$ splits as a direct sum of line bundles, $E = \mathcal{O}(a_1) \oplus \dots \oplus \mathcal{O}(a_r)$. Given a vector bundle $E \rightarrow \mathbb{P}^m$, we let the *splitting type* of E denote the sequence of integers (a_1, \dots, a_r) one obtains by restricting E to a general $\mathbb{P}^1 \subset \mathbb{P}^m$. We say E is *uniform* if the sequence is independent of the $\mathbb{P}^1 \subset \mathbb{P}^m$.

$E := \phi(\mathcal{O}_{\mathbb{P}A} \otimes V)$ is uniform of splitting type $E|_{\mathbb{P}^1} = \mathcal{O}^{\frac{r}{2}} \oplus \mathcal{O}^{\frac{r}{2}}(1)$, because E is globally generated and $E \simeq E^*(1)$. This proves 8.3..iv. (8.3..iv. can also be proved using a normal form, see [HP].)

8.5.2 Examples.

$$E_{G(2,2k+1)^*} = T\mathbb{P}^2(-1)^{\oplus k}$$

$$E_{\text{Seg}(\mathbb{P}^1 \times \mathbb{P}^n)^*} = \text{split}$$

$$E_{\mathbb{S}_5} = \Lambda^2(T\mathbb{P}^4(-1))$$

If A is the doubling of B , then $E_A = E_B \oplus E_B^*(1)$. (This makes sense because $\mathbb{P}A \simeq \mathbb{P}B$.)

If A is the doubling of a suitably generic $B^{\frac{r}{2}} \subset \mathbb{C}^{\frac{r}{2}} \otimes \mathbb{C}^{m-\frac{r}{2}}$, then $E_A = \mathcal{O}^{\frac{r}{2}} \oplus \mathcal{O}^{\frac{r}{2}}(1)$, justifying the terminology “split type”.

Relating the systems to dual varieties, we have $\mathbb{P}A = \mathbb{P}N_H^*X^* = \mathbb{P}^{\delta*}$, $E = N_{X/Y_H}^*$. Ein observed that $E \simeq E^*(1)$, but he failed to observe that the identification arises from a symmetric map, although it is implicit in his proof.

We actually proved a more general result than [IL, 2.16], namely:

8.5.3 Theorem [IL,1.2]. *Let Z be a nonsingular simply connected projective variety of dimension δ , E a rank m vector bundle on Z , and L a line bundle on Z . Suppose that $S^2(E^*) \otimes L$ is an ample vector bundle and that there is a constant even rank $r \geq 2$ symmetric bundle map $E \rightarrow E^* \otimes L$. Then $\delta \leq m - r$.*

Idea of proof. Consider the case that $Z \subset \mathbb{P}S^2V^*$ is a variety of quadrics. Consider the incidence correspondence

$$(8.5.4) \quad \mathcal{I} = \{(x, P) \mid x \in P\} \subset \text{Seg}(\mathbb{P}V \times Z) \subset \mathbb{P}(V \otimes S^2V^*)$$

\mathcal{I} is the intersection of the Segre with a cubic hypersurface, and thus we can use the Lefschetz theorem to conclude that $h^i(\mathcal{I}) = h^i(\mathbb{P}^m \times Z)$ for $i < m + \delta - 1$. By the Künneth formula, $h^i(\mathbb{P}^m \times Z) = \bigoplus_{l+s=i} h^l(\mathbb{P}^m)h^s(Z)$. \mathcal{I} can be considered as a fibration $Q \rightarrow \mathcal{I} \rightarrow Z$. Since $\pi_1(Z) = 0$ there is no monodromy and were the fiber smooth, the Leray spectral sequence would say that we could still compute the cohomology as if it were a product, i.e. $h^i(\mathcal{I}) = \bigoplus_{l+s=i} h^l(Q)h^s(Z)$. If we had the above results, we could conclude $h^{r-2}(Q) = 2 \neq 1 = h^{r-2}(\mathbb{P}^m)$ and unless $\delta \leq m - r$ there would be a contradiction.

We can easily modify things to make the fiber smooth, we let \mathcal{I}' be the fiber bundle over Z with fiber the smooth quadric obtained by quotienting Q by its singular locus, and note that \mathcal{I}' is contained in a projective bundle whose fibers are $\mathbb{P}(V/Q_{z,sing})$. Now Leray applies to \mathcal{I}' but unfortunately Lefshetz in its standard form no longer applies when comparing \mathcal{I}' and this vector bundle. Fortunately a version of the Lefschetz theorem due to Lazarsfeld does apply in this situation and one obtains the result. \square

8.5.5 Problem. *Classify the boundary cases $\delta = m - r$.*

[IL,1.2] is an analogue for symmetric matrices of the following result of Lazarsfeld:

8.5.6 Theorem [Laz]. *Let X be a projective variety of dimension m . Let E and F be vector bundles on X of ranks e and f respectively. Suppose that $E^* \otimes F$ is ample and that there is a constant rank r vector bundle map $E \rightarrow F$. Then $m \leq e + f - 2r$.*

8.3.i.1 follows from (8.5.6). What follows is a simplified version of Westwick’s proofs of 8.3.i.1 and 8.3.i.2, due to B. Ilic:

Proof of i.1 and the remainder of of i.2. let K denote the kernel bundle and N the cokernel bundle of ϕ . From the resulting exact sequences, $c(K)c(E) = 1$ and $c(E)c(N) = (1+h)^n$. Thus, $c(K)(1+h)^n = c(N)$. If $n - r + 1 \leq i \leq l$ then $c_i(N) = 0$ and looking at the coefficient of h^i we get $\sum_{j=0}^{m-r} \binom{n}{i-j} c_j(K) = 0$ where we use the convention that $\binom{n}{j} = 0$ if $j < 0$ or $j > n$. The coefficient matrix of this collection of linear equations is

$M = ((\binom{n}{i-j}))_{0 \leq j \leq m-r, n-r+1 \leq i \leq l}$. If $l = m + n - 2r + 1$ then this is a square invertible matrix with determinant $\prod_{j=0}^{m-r} j!$. Thus $k_0 = 0$ which is a contradiction since $k_0 = 1$. This proves i.1. Westwick refers to a privately published manuscript of Muir and Metzler as a reference for evaluating this determinant however one can also refer to e.g. [ACGH, pg. 93-95]. i.2 follows directly from considering the linear equation with $i = n - r + 1$. \square

8.6 Systems of split type.

In this section we restrict attention to systems of quadrics, the modifications for the skew case being more or less clear.

8.6.1 Theorem (Sato) [Sato]. *Let $E^r \rightarrow \mathbb{P}^\delta$ be a uniform vector bundle. If $\delta > r$, then E splits as a direct sum of line bundles. If $\delta = r$ then either E splits or E is isomorphic to $T\mathbb{P}^\delta$ or $T\mathbb{P}^{\delta*}$.*

The proof of Saito's theorem relies on Tango's result about maps of projective spaces to Grassmannians (which follows from an elementary Chern class calculation) and sheafy gymnastics.

We will call systems of quadrics whose associated vector bundle E splits, *of split type*.

8.6.2 Conjecture. *If $E^r \rightarrow \mathbb{P}^\delta$ is a vector bundle arising from a system of quadrics and if $\delta > \frac{r}{2} + 1$, then E splits as a direct sum of line bundles.*

Note that this is the best bound possible due to the system $I_2(G(2,5))$. I see no reason for the corresponding conjecture to be true in the skew case, as the corresponding Lagrangian Grassmannian is much larger than the Spinor variety.

To study this conjecture let's examine maps of projective spaces into spinor varieties as follows (Chern classes are not enough):

Let $A \subset \mathbb{P}S^2T^*$ be a system of constant rank. Fix $Q_0 \in A$. Consider $\mathbb{P}(A/A_0) = \mathbb{P}^{\delta-1} = \mathbb{F}(Q_0)$, the space of lines in A through Q_0 . There is a natural vector bundle L over $\mathbb{F}(Q_0)$, where

$$(8.6.3) \quad L_{[Q]} = \text{span}_{\{[s,t] \in \mathbb{P}^1\}} \ker(sQ + tQ_0) \bmod \ker Q_0.$$

Since all pencils $\{Q, Q_0\}$ are of the same type, L is indeed a vector bundle and not just a sheaf.

Note that L is a subbundle of the trivial bundle $\mathcal{O}^r = \mathcal{O} \otimes (V/\ker Q_0)$. It is of rank $\frac{r}{2}$ (see [HP]). Label the resulting exact sequence:

$$(8.6.4) \quad 0 \rightarrow L^{\frac{r}{2}} \rightarrow \mathcal{O}_{\mathbb{F}(Q_0)}^r \rightarrow \Lambda^{\frac{r}{2}} \rightarrow 0$$

$L_{[Q]}$ is a Q_0 -isotropic subspace of $(V/\ker Q_0)$ since $L_{[Q]} + \ker Q_0 = W_{\{Q, Q_0\}}$. Thus the image of the map to the Grassmannian $G(\frac{r}{2}, V/\ker Q_0)$ defined by L actually lies in the Q_0 -spinor variety, \mathbb{S}_{Q_0} , a connected component of $\{E \in G(\frac{r}{2}, (V/\ker Q_0)) \mid E \subset Q_0\}$. Label the resulting map

$$(8.6.5) \quad \begin{aligned} \phi : \mathbb{P}^{\delta-1} &\rightarrow \mathbb{S}_{Q_0} \\ [Q] &\mapsto L_{[Q]} \end{aligned}$$

$L = \Lambda^*$, as $L = \phi^*(\mathcal{S})$, $\Lambda = \phi^*(\mathcal{Q})$ where \mathcal{S}, \mathcal{Q} denote the pullbacks of the universal sub- and quotient bundles on the Grassmannian to \mathbb{S} , and $\mathcal{S} = \mathcal{Q}^*$ when pulled back to \mathbb{S} . This proves the assertion $\text{rank } L = \frac{r}{2}$. (The isomorphism is skew symmetric because $T_E \mathbb{S} \simeq \Lambda^2 E$.)

To prove the conjecture, one would need to show that if $\frac{r}{2} < \delta - 1$, then L is trivial, i.e. ϕ maps to a point (as then $W_{\{Q, Q_0\}}$ is independent of Q).

8.6.6 Proposition. *If $k > 1$, there are no nonconstant maps $\mathbb{P}^k \rightarrow \mathbb{S}$ or $\mathbb{P}^k \rightarrow \mathbb{G}_{Lag}$ such that $\phi^*(\mathcal{S})$ is a split vector bundle, where \mathcal{S} denotes the universal subbundle on the Grassmannian restricted to \mathbb{S} (resp. \mathbb{G}_{Lag}).*

Thus it would be sufficient to show that L must be uniform, because if L were uniform in this range, it would have to be split.

Proof of proposition. Given a map ϕ , one gets a sequence as (8.6.4). If L splits we may write

$$(8.6.7) \quad \begin{aligned} c(L) &= \prod_{i=1}^{\frac{r}{2}} (1 + \alpha_i t) \quad \alpha_i \leq 0 \\ c(\Lambda) &= \prod_{i=1}^{\frac{r}{2}} (1 - \alpha_i t) \end{aligned}$$

We have

$$1 = c(L)c(\Lambda) = 1 - \{\Sigma_i (\alpha_i)^2\}t^2 + \dots$$

which implies $\alpha_i = 0$ for all i and thus L is trivial. \square

The conjecture would also follow from the following stronger conjecture:

8.6.8 Conjecture. *If $p < q$ then there are no nonconstant maps, $\mathbb{P}^p \rightarrow \mathbb{S}_q$*

For the equality case $p = q$ there are linear examples. One might hope to prove the conjecture by a degeneration argument.

8.7 Split type systems and dual varieties.

8.7.1 Proposition. *If $X^n \subset \mathbb{P}^{n+a}$ is a linear fibration with $f > \frac{n}{2}$ dimensional fibers, then $\delta_* \geq 2f - n$*

It seems reasonable to conjecture that split type systems correspond to fibrations with linear fibers. This cannot be quite correct, as if $\delta_* = 1$, then $|II_{X^*, H}|$ is necessarily of split type.

Fix $H = [e^{n+a}]$. Let $N' \subset N_x X$ be $(e^{n+a})^\perp$ and let $L = \hat{q}_{sing}^{n+a} \subset T$. Consider the linear map

$$(8.7.2) \quad \begin{aligned} T/L &\rightarrow L^* \otimes N' \\ v &\mapsto (w \mapsto II(v, w)) \end{aligned}$$

which is well defined, as since q^{n+a} is generic, its singular locus is in the base locus of the system.

8.7.3 Theorem [L10]. *If $II_{X^*,H}$ is of split type and (8.7.2) is injective, then X is a linear fibration with $\frac{n+\delta_*}{2}$ -dimensional fibers.*

For the proof, see [L10]. Note that if δ_* is large and X is smooth, then (8.7.2) is automatically injective.

Note that Ein's fibration theorem agrees with this conjecture, using the standard bound. That is, in the geometric situation, $r = n - \delta_*$ (and $\delta_* = \delta$), so Ein proves the variety must be a fibration in the range $|II|_{X^*,H}$ is known to be split type. If our conjecture is correct, then combined with the other results, this would say that if $\delta_* > \frac{n+1}{3}$, then X must be a fibration.

The fibration conjecture combined with [RVdV] would imply that $\delta_* > \frac{n+2}{3}$ implies $a > \frac{2}{3}n + 1$.

8.8 “Dual” systems of quadrics.

This section contains some observations that should have geometric consequences, although I have not yet been able to determine them.

Let T^n, N^a be vector spaces and let $II \in S^2 T^* \otimes N$ be a system of quadrics on T parametrized by N of bounded rank r . Given a generic $\alpha \in N^*$, let $W = W_\alpha := \text{singloc } II^*(\alpha)$. Bertini's theorem implies that $W \subset \text{Base } |II|$ so for each $\beta \in N^*$, the map

$$(8.8.1) \quad II_\beta^* : T \rightarrow T^*$$

descends to a well defined mapping

$$(8.8.2) \quad II_\beta^{*\prime} : W \rightarrow W^\perp$$

i.e. each generic $\alpha \in N^*$ gives a tensor

$$(8.8.3) \quad L'_\alpha \in (N^*/\{\alpha\}) \otimes W_\alpha \otimes W_\alpha^\perp$$

which we consider as a system of linear forms L_α on $(N^*/\{\alpha\}) \otimes W_\alpha^\perp$ parametrized by W_α . Call this system the system *induced by α* .

What is remarkable is that the doubling of the induced system occurs naturally in the geometric setting. More precisely, examining the inversion formula ([IL], 3.9), one has the following proposition:

8.8.4 Proposition. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety with degenerate dual variety X^* of dimension $r + a - 1$, let $x \in X_{sm}$, let $H \in X_{sm}^*$ be a hyperplane tangent at x and let $n_H \in N_x^* X$ be a vector representing H . Then $II_{X^*,H}$ modulo the contributions of $\partial II_{X,x}$ is the doubling of the system induced by n_H . (After one makes identifications as in [IL], 3.9).*

This places further, to my knowledge so far unstudied, restrictions on the second fundamental forms of varieties with degenerate dual varieties. For example the system induced by n_H must be of bounded rank $\frac{r}{2}$.

An interesting example is when X is self-dual, as in that case, for all $\alpha \in N$, the doubling of the induced system is the original system. One might hope to characterize all such systems of quadrics.

§9. VARIETIES WITH DEGENERATE SECANT AND TANGENTIAL VARIETIES

9.1 Examples. The only examples of smooth varieties with degenerate secant varieties that I am aware of essentially come from varieties of rank one matrices. It's clear that the secant variety will be the set of rank less than or equal to two matrices, which often has a secant defect. Consider the basic examples

- i. Segres, $\mathbb{P}^m \times \mathbb{P}^n$, $n \geq m$, $\delta_\sigma = 2$ if $n \geq 2$.
- ii. Veroneses (symmetric matrices) $v_2(\mathbb{P}^n)$, $\delta_\sigma = 1$ if $n \geq 2$.
- iii. Grassmannians of two-planes (skew symmetric matrices) $G(2, n)$, $\delta_\sigma = 4$ if $n \geq 6$.
- iv. Severi varieties, \mathbb{A} -Hermitian symmetric 3×3 matrices. $\mathbb{A}\mathbb{P}^2$, $\delta_\sigma = \dim_{\mathbb{C}} \mathbb{A}$

There is no known example of a variety with $\delta_\sigma > 8$.

In what follows, I only mention secant or tangential varieties in a hypothesis, with the corresponding statement for the other understood.

9.2 Theorems on Secant varieties. Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety not contained in a hyperplane.

- i. (Zak's Theorem on Linear Normality, [FL],[Z]) If $\sigma(X) \neq \mathbb{P}^{n+a}$, then $a \geq \frac{n}{2} + 2$.
- ii. (Zak's Theorem on Severi Varieties, [LV],[Z]) If $a = \frac{n}{2} + 2$, and $\sigma(X) \neq \mathbb{P}^{n+a}$, then X is one of the four Severi varieties $\mathbb{A}\mathbb{P}^2 \subset \mathbb{P}\mathcal{H}$.
- iii. (Zak-Fantecchi superadditivity Theorem [Z],[Fan] [L5])

$$\dim \sigma_k(X) \leq n + (k-1)(n+1 - \delta_\sigma).$$

iv. ([Roberts], [L5]) Let $Y \subset \mathbb{P}V$ be a variety and let $X = v_d(Y) \subset \mathbb{P}S^dV$ be the Veronese re-embedding. If $d > 2$ or $d = 2$ and Y is not a linear subspace of $\mathbb{P}V$, then $\sigma(X)$ is nondegenerate.

v. [Zak, V.2.3] Let $M(n, \delta)$ denote the maximum ambient dimension of a smooth variety of dimension n , secant defect δ . Then $M(n, \delta) \leq \frac{n+\delta+2+\epsilon(\delta-\epsilon-2)}{2\delta}$ where ϵ is the remainder of the division of n by δ .

vi. [GH] $\delta_\gamma(\tau(X)) \geq \delta_\tau(X) + 1$. (Note that $\delta_\gamma(\tau(X)) \geq 1$ for any tangential variety.)

vii. [Zak] $\delta_*(\tau(X)) \geq \delta_\sigma(X) + 1$

The Severi varieties have many other special properties. For example, they classify the quadro-quadro Cremona transforms (see [ESB]). These four varieties turn up in several areas of geometry to construct examples of varieties exhibiting extremal pathology (isoparametric submanifolds [C, III.1 p1447], tight embeddings [Kuiper]).

The $d > 2$ case of iv. is due to Roberts, the $d = 2$ to myself.

9.3 The refined third fundamental form.

For $v \in T$ and $A \subset S^2 T^*$, recall the notions

$$(9.3.1) \quad \begin{aligned} \text{Ann}(v) &:= \{q \in A \mid [v] \in q_{\text{sing}}\} \\ \text{singloc}(A) &:= \{[v] \in \mathbb{P}T \mid [v] \in q_{\text{sing}} \forall q \in A\} \end{aligned}$$

Recall the map $\gamma^{(2)} : X \rightarrow G(n+a_1, V)$ defined in (*). If $II(v, T) \subsetneq II(T, T)$, $\gamma^{(2)}$ can be refined to a map

$$(9.3.2) \quad \begin{aligned} \tilde{\gamma}^{(2)} : TX &\dashrightarrow G(n+a_0, V) \\ (x, v) &\mapsto \hat{T} + \widehat{II}(v, T) \end{aligned}$$

Let $v \in T = T_x X$ and let II_v denote the mapping:

$$(9.3.3) \quad \begin{aligned} II_v : T &\rightarrow N \\ w &\mapsto II(v, w) \end{aligned}$$

Taking the derivative of $\tilde{\gamma}^{(2)}$, we obtain a refinement of the third fundamental form: In place of $V/\hat{T}^{(2)}$ we get the larger space $V/(\hat{T} + II_v(T)) \simeq \text{Ann}(v)^*$. In place of T , we must restrict to $SA(v) := \text{singloc Ann}(v)$, in order to have a mapping symmetric in the three factors.

Fix a II -generic vector $v \in T$, define the *third fundamental form refined with respect to v* ,

$$(9.3.4) \quad III^v \in S^3(\text{singloc}(\text{Ann}(v)))^* \otimes N/II_v(T)$$

III^v is well defined as if $w \in \text{singloc}(\text{Ann}(v))$, then $II_w(T) \subseteq II_v(T)$.

To study III^v , it is natural to consider the frame bundle over general points $(x, v) \in TX$, and for each such, we get a refinement of the flag $\hat{x} \subset \hat{T} \subset V$ to

$$\hat{x} \subset \{\hat{x}, v\} \subset \hat{T} \subset \{\hat{T} + II(v, v)\} \subset \{\hat{T} + II(v, T)\} \subset V$$

In fact we may refine to a flag

$$\hat{x} \subset \{\hat{x}, v\} \subset \{\hat{x}, SA(v)\} \subset \hat{T} \subset \{\hat{T} + II(v, v)\} \subset \{\hat{T} + II(v, T)\} \subset V$$

in fact to a partially ordered flag, because we also have the flag

$$\{\hat{x}\} \subset \{\hat{x}, \ker II_v\} \subset \{\hat{x}, SA(v)\} \subset \hat{T} \subset \{\hat{T} + II(v, v)\} \subset \{\hat{T} + II(v, T)\} \subset V$$

One can recover III from knowing III^v for all $v \in T$.

9.4 Theorems on the local differential geometry of varieties with degenerate secant varieties. Let $X^n \subset \mathbb{P}^{n+a}$ be a open subset of a variety with degenerate tangential manifold of dimension $n+a_0$.

i. (Terracini ??) [GH] [L5] Let $X^n \subset \mathbb{P}V$ be a smooth variety. let $x \in X$ be a general point and $v \in T_x X$ a generic tangent vector. Then

$$\dim \tau(X) = n + \dim II_v(T)$$

In particular, if $a \leq n$, then $\delta_\tau = \dim \ker II_v$.

ii. Terracini ?? [GH] Let $X^n \subset \mathbb{P}V$ be a smooth variety, let $x \in X$ be a general point. Let $v \in T = T_x X$ be II -generic, then

$$\dim \sigma(X) = \begin{cases} n + \dim II_v(T) & \text{if } III^v(v, v, v) = 0 \\ n + \dim II_v(T) + 1 & \text{if } III^v(v, v, v) \neq 0 \end{cases}.$$

iii. $III \neq 0$ implies $\sigma(X)$ is nondegenerate.

iv. Rank restrictions for varieties with degenerate tangential varieties, [L5, 7.1] Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety with degenerate secant variety that is a hypersurface. Let $x \in X$ be a general point. Let r_τ be the maximum rank of a quadric in $|II|_x$ annihilating a II -generic vector in $T_x X$. Then

$$r_\tau \geq n - a + 2.$$

9.4.1 Exercise. Show directly that if $a > \binom{n+1}{2}$, then $\delta_\sigma(X) = 0$.

9.4.2 Exercise. Show that the condition $\dim II_v(T) = a_0$ is equivalent to showing the image of the rational map

$$\begin{aligned} ii : \mathbb{P}T &\dashrightarrow \mathbb{P}N \\ [v][II(v, v)] \end{aligned}$$

has an a_0 -dimensional image, or, in other words, that the quadrics in $|II|$ satisfy $a - a_0$ polynomial equations.

The proof of i. is given below. To calculate $\dim \sigma(X)$ one takes a general point of $\sigma(X)$ written as the sum of two points on X and calculates the tangent space by expanding a Taylor series around one of the points. See [L5 §10] for details.

vi. is a stronger rank restriction than for arbitrary varieties in two ways: first, the rank is higher, and second, one looks not at generic quadrics, but at generic quadrics in the subvariety of quadrics annihilating a II -generic vector. Note that in vi, $\delta_\tau + r_\tau \leq n$ with equality only in the borderline case of Zak's theorem on linear normality.

To prove vi, one studies the subvariety $X_\Delta^* \subset X^*$ of hyperplanes H such that at some point of tangency, H annihilates a II -generic vector. One first concludes that over a general point $x \in X$, $\dim (X_\Delta^* \cap \mathbb{P}N_x^* X) \geq r_\tau$, and then shows that in fact equality occurs.

Each $H \in X_\Delta^*$ is tangent to a δ_τ -dimensional subvariety Y_H of X whose tangent space at x is $(q_H)_{\text{sing}}$. In the case of the Severi varieties these subvarieties Y_H are quadric hypersurfaces in a linear space. They were exploited in Zak's classification Severi varieties.

9.5 Dimension of $\tau(X)$.

The dimension of $\tau(X)$ is the dimension of its tangent space at a smooth point. We work on \mathcal{F}^1 , which we consider as a bundle over the smooth points of $\tau(X)$ by the mapping

$$\begin{aligned} (9.5.1) \quad \mathcal{F} &\rightarrow \tau(X) \\ f &\mapsto [e_1]. \end{aligned}$$

We may take $[e_1]$ as a typical element of $\tau(X)$. Let $2 \leq \rho, \sigma \leq n$. By (*),

$$(9.5.2) \quad de_1 \equiv \omega_1^0 e_0 + \omega_1^\rho e_\rho + \omega_1^\mu e_\mu \pmod{[e_1]}$$

and $\dim T_{[e_1]} \tau(X) = \{ \text{the number of independent 1-forms in (9.5.2)} \}$. Over X , $\{\omega_1^0, \omega_1^\rho\}$ are independent so we only need to know the number of independent one-forms among the ω_1^μ , but (up to twisting)

$$(9.5.3) \quad \omega_1^\mu \otimes (e_\mu \bmod \hat{T}) \equiv II_{X,x}(\underline{e}_1, \cdot).$$

where $\underline{e}_1 = e_0^* \otimes (e_1 \bmod e_0) \in T^*$. (9.5.3) proves i.

9.6 Towards an infinitesimal version of the connectedness theorem.

Note that we can compute $II_{\tau(X), [e_1]}$ from third order information at $x = [e_0]$. Adapt further such that $II_{X, \underline{e}_1}(T) = \{e_{n+h}\}$. Let $1 \leq \lambda \leq a_0$, $a_0 + 1 \leq \epsilon \leq n$ and adapt such that $\ker(II_{\underline{e}_1}) = \{e_\epsilon\}$. Then

$$(9.6.1) \quad II_{\tau(X),[e_1]} = (r_{1\lambda k}^{n+h} \omega_0^\lambda + 2q_{jk}^{n+h} \omega_1^j) \omega_0^k \otimes e_{n+h}$$

$e_\epsilon \in \text{singloc } II_{\tau(X),[e_1]}$, which proves 9.2.vi.

If X is smooth, the connectedness theorem implies $III^v = 0$. In frames, if we let, $\text{singloc}(\text{Ann}(e_1)) = \{e_1, e_s, e_\epsilon\}$, then $III^v = 0$ implies further that

$$(9.6.2) \quad r_{11k}^{n+h} = r_{1sk}^{n+h} = 0.$$

Griffiths and Harris refer to (9.2.vi) as the “deepest result” in their paper. They speculate that it should be related to the connectedness theorem. (9.4.iv) is valid for an open subset X^0 of any variety X , but $\overline{\tau(X^0)} \neq \tau(X)$ in general, essentially because one uses the tangent star instead of the tangent cone to define $\tau(X)$. See [L5] for a discussion. Thus there is no hope for a purely local version of the Fulton-Hansen theorem (3.6).

On the other hand, one might hope to prove directly that if X is smooth, then $III^v = 0$ holds solely based on smoothness considerations or *a priori* restrictions on F_3 present when II takes a certain form (see §13). For example, it would be sufficient to prove that $|II|^{(1)} = 0$.

Such a result would give a proof of (3.6) that is local in nature in the case X is smooth. Examples of second fundamental forms where the additional vanishing (9.6.2) is automatic are given in [L9].

§10. SYSTEMS OF QUADRICS WITH TANGENTIAL DEFECTS

The main result of the second step in my study of the infinitesimal geometry of varieties with degenerate secant varieties is the following:

10.1 Theorem [L5]. *Let $|II| \subset \mathbb{P}S^2T^*$ be a system of quadrics arising from the second fundamental form at a general point of a smooth variety $X^n \subset \mathbb{P}^{n+a}$, $a \leq n$ having a degenerate secant variety that is a hypersurface.*

Let $v \in T$ be a generic tangent vector. Consider the mapping $II_v : T \rightarrow N$, defined by $II_v(w) = II(v, w)$. Let $P \in |II|$ denote the unique quadric such that $[v] \in P_{\text{sing}}$. Then

i. $\ker II_v \subset P_{\text{sing}}$.

ii. If $\{v, \ker II_v\} = P_{\text{sing}}$, then there exists a canonical (possibly degenerate) quadratic form Q_v on $\ker II_v$.

iii. Under the hypotheses of ii., T/P_{sing} is a Clifford($\ker II_v, Q_v$) module.

In other words, in this case, each smooth point of $\sigma(X)$, produces a canonical Clifford($\mathbb{C}^{\delta_\sigma}$) action on a vector space of dimension less than $n - \delta_\sigma$.

Note that assuming $\sigma(X)$ is a hypersurface is no loss of generality because one can always project until $\sigma(X)$ is a hypersurface. The conclusions of ii, iii are valid in slightly more general contexts.

10.2 Definition Let T be an n -dimensional vector space and let $A \subset S^2T^*$ be an a -dimensional system of quadrics, with $a \leq n$. We say A has a *tangential defect* of δ_τ if for all $v \in T$, there exists a δ_τ -dimensional subspace $U_v \subset A$ such that for all $q \in U_v$, $[v] \in q_{\text{sing}}$, i.e. $q(v, w) = 0$ for all $w \in T$.

Equivalently, consider the rational map:

$$\begin{aligned} ii : \mathbb{P}T &\dashrightarrow \mathbb{P}A \\ x &\mapsto [Q^1(x), \dots, Q^a(x)] \end{aligned}$$

then $\delta_\tau = \text{codim } ii(\mathbb{P}T)$. To see the equivalence of the two definitions, note that II_v is the derivative of ii at $[v]$ after taking appropriate quotients.

It is often useful to consider a parametrized system of quadrics to mimic the geometric situation. We will write $II \in S^2 T^* \otimes N$ where N is an a -dimensional vector space, and $|II| \subset \mathbb{P}S^2 T^*$ denote the corresponding system.

For simplicity, we often assume II corresponds to the second fundamental form at a point of a variety X where $\sigma(X)$ is a hypersurface. In this case, for generic $v \in T$, $q = \text{Ann}(v)$ is unique. Call this a *critical tangential defect*. Note that for a critical tangential defect, $\delta_\tau = n - a + 1$.

In the case of a critical tangential defect, for any II -generic $v \in T$, there is a natural hyperplane in $|II|$, namely $II(v, v)^\perp$.

The following proposition is analogous to the fact that $\hat{x} \subset \ker d\gamma_x(v)$ for all $v \in T$:

10.3 Proposition. $\ker II_v \subset \text{singloc } \text{Ann}(v)$

The proof follows from the notion of the projective second fundamental form of a mapping, see [L5].

10.3.1 Exercise. Prove (10.3) using moving frames.

Systems of quadrics with a tangential defect satisfy a Bertini type theorem. Recall that the classical Bertini theorem says that if $A \subseteq S^2 T^*$ is any system of quadrics (or polynomials of any degree for that matter) and $q \in A$ is a general element, then $q_{\text{sing}} \subseteq \text{Base}(A)$. We can't hope for such a strong statement for $q = \text{Ann}(v)$, because $v \notin \text{Base}|II|$. But in fact the next best thing is true.

10.4 Bertini type Lemma, [L5, 6.16]. *Let $II \in S^2 T^* \otimes N$ be a system of quadrics with a critical tangential defect. Let $v \in T$ be II -generic, let $q \in |II|$ be the annihilator of v . Then $q_{\text{sing}} \subseteq \text{Base}(II(v, \hat{q}_{\text{sing}})^\perp)$.*

This Bertini type lemma implies that $|II|/II(v, \hat{q}_{\text{sing}})^\perp$ can be considered as a system of quadrics on \hat{q}_{sing} . In the case of a critical defect, we obtain a well defined quadric which we denote \tilde{Q}_v .

Note that there are large linear subspaces in $\text{Base}|II|$ because any vector in the base locus of \tilde{Q}_v must be (as an element of $\mathbb{P}T$) in the base locus of $|II|$. Moreover, there is a positive dimensional variety of different \tilde{Q}_v 's.

Fix $v \in T$, II -generic. We have the subspaces $\ker II_v$ and $\text{singloc}(\text{Ann}(v))$ giving a flag

$$(10.5) \quad \ker II_v \subset \{v, \ker II_v\} \subseteq \text{singloc}(\text{Ann}(v)) \subset T$$

For future reference, we record here the way these spaces will be referred to in indices:

$$(10.6) \quad \{e_\epsilon\} \subset \{e_1, e_\epsilon\} \subseteq \{e_1, e_\epsilon, e_s\} = \{e_\xi\} \subseteq \{e_1, e_\epsilon, e_s, e_j\} = \{e_\alpha\}$$

I.e. $\{e_\epsilon\} = \ker II_v$, $\{e_s\} = \text{singloc}(\text{Ann}(v)) / \{v, \ker II_v\}$, $\{e_j\} = T / \text{singloc}(\text{Ann}(v))$.

10.7 Theorem, the canonical Clifford algebra structure, [L5, 6.23]. *Let $A \subset S^2 T^*$ be a system of quadrics with a critical tangential defect, and let $v \in T$ be II -generic. Assume $|II|/II(v, \hat{q}_{sing})^\perp$ restricted to $\ker II_v$ corresponds to a single quadratic form which we denote Q_v . Then $T/(Ann(v))_{sing}$ is canonically a $Clifford(\ker II_v, Q_v)$ module.*

10.8 Remark/question. In the cases Q_v exists, it is likely its rank is linearly bounded from below, and thus in this situation, the tangential defect δ_τ could grow at best logarithmically with respect to $\dim X$. This is because $\dim \ker II_v = \delta_\tau$ and $n > \dim(T/(singloc(Ann(v))))$. Were Q_v nondegenerate, then $\dim(T/(singloc(Ann(v))))$ would have to be on the order of $2^{\delta_\tau - 1}$.

Proofs. Before proving (10.4), let's warm up by proving the classical Bertini lemma for an arbitrary system of quadrics. To keep notation consistent for what comes, I'll assume the system of quadrics occurs as the second fundamental form at some point of a open subset of a variety. Writing the system as $\{\omega_\alpha^\mu \omega_0^\alpha = q_{\alpha\beta}^\mu \omega_0^\alpha \omega_0^\beta\}$, assume q^{n+a} is a generic quadric in the system (further indices, $n+1 \leq \mu, \nu \leq n+a, 1 \leq \lambda \leq a-1$) and that $\mathbb{P}\{e_\xi\} = q_{sing}^{n+a}$, $1 \leq \xi, \eta \leq \dim q_{sing}^{n+a}$. Differentiating, we have

$$(10.9) \quad r_{\xi\eta\beta}^{n+a} \omega_0^\beta = q_{\xi\eta}^{n+\lambda} \omega_{n+\lambda}^{n+a}$$

Since we have assumed q^{n+a} is a generic quadric, the forms $\omega_{n+\lambda}^{n+a}$ are all linearly independent and independent of the semi-basic forms ω_0^β . Thus both sides of the equation above have to vanish and hence the coefficients $q_{\xi\eta}^{n+\lambda}$ are zero, proving the classical Bertini theorem for a system of quadrics.

Proof of 10.3. For simplicity assume we have a critical tangential defect, so that for a II -generic vector $v \in T$, $Ann(v)$ consists of a single quadric. We adapt frames such that $e_1 = v$ and $q^{n+a} = Ann(v)$. Since q^{n+a} is no longer a generic quadric in the system, the forms $\omega_{n+\lambda}^{n+a}$ are no longer necessarily independent or independent of the semi-basic forms.

Let $[e_j]$ denote a basis of T/q_{sing}^{n+a} , $2 \leq j, k, l \leq a-1$. Since we have a critical defect, $II_v : T \rightarrow N$ has rank $a-1$ and $\text{rank}(q^{n+a}) = a-2$. We may normalize further such that $II(v, v) = e_{n+1}$, i.e. that $e_1 \in \text{Base}\{q^{n+k}, q^{n+a}\}$, and in fact that $II(\underline{e}_1, \underline{e}_s) = \underline{e}_{n+s}$, $II(\underline{e}_1, \underline{e}_j) = \underline{e}_{n+j}$. We need to show that $q_{sing}^{n+a} \subset \text{Base}\{q^{n+k}, q^{n+a}\}$,

Computing we have

$$(10.10) \quad r_{11\beta}^{n+a} \omega_0^\beta = \omega_{n+1}^{n+a}$$

$$(10.11) \quad r_{1j\beta}^{n+a} \omega_0^\beta = -\omega_{n+j}^{n+a} + q_{jk}^{n+a} \omega_1^k$$

$$(10.12) \quad r_{1s\beta}^{n+a} \omega_0^\beta = -\omega_{n+s}^{n+a}$$

Now for the essential point: Since e_1 is generic and the e_k were taken generically, the forms ω_1^k are linearly independent and independent of the semi-basic forms ω_0^β . Moreover, the matrix q_{jk}^{n+a} is invertible. From (10.10), (10.13) we see that $\omega_{n+1}^{n+a}, \omega_{n+s}^{n+a}$ are semi-basic. (10.11) implies we can solve for ω_{n+j}^{n+a} in terms of the ω_1^k modulo the semi-basic forms, so

the forms ω_{n+j}^{n+a} are independent and independent of the semi-basic forms. Now consider (10.9) modulo the basic forms. We have

$$(10.13) \quad 0 \equiv q_{\xi\eta}^{n+j} \omega_{n+j}^{n+a} \bmod \{\omega_0^\alpha\}$$

(where we are still using e_ξ as a basis for q_{sing}^{n+a} although index ranges have been shifted).

(10.3) follows as the forms ω_{n+j}^{n+a} are all independent and independent of the semi-basic forms. \square

Proof of the Clifford algebra structure. Assume we are in the case $\{v, \ker II_v\} = \text{singloc}(\text{Ann}(v))$. (This is the case if $a = \frac{n}{2} + 2$.) Consider

$$(10.14) \quad \begin{aligned} r_{11\beta}^{n+j} \omega_0^\beta &= -\omega_{n+1}^{n+j} + 2\omega_1^j, \\ r_{1\epsilon\beta}^{n+j} \omega_0^\beta &= \omega_\epsilon^j + q_{\epsilon k}^{n+j} \omega_1^k. \end{aligned}$$

which imply

$$(10.15) \quad \begin{aligned} \omega_{n+1}^{n+j} &\equiv 2\omega_1^j \bmod \{\omega_0^\alpha\}. \\ \omega_\epsilon^j &\equiv -q_{\epsilon k}^{n+j} \omega_1^k \bmod \{\omega_0^\alpha\}. \end{aligned}$$

Computing

$$(10.16) \quad r_{\epsilon\delta\beta}^{n+j} \omega_0^\beta = -q_{\epsilon\delta}^{n+1} \omega_{n+1}^{n+j} + q_{\epsilon k}^{n+j} \omega_\delta^k + q_{\delta k}^{n+j} \omega_\epsilon^k$$

and moding out by the semi-basic forms and using (10.15), we obtain,

$$(10.17) \quad q_{\epsilon k}^{n+j} (-q_{\delta i}^{n+\kappa} \omega_1^i) + q_{\delta k}^{n+j} (-q_{\epsilon i}^{n+\kappa} \omega_1^i) \equiv q_{\epsilon\delta}^{n+1} (2\omega_1^j) \bmod \{\omega_0^\alpha\}.$$

I.e. that

$$(10.18) \quad q_{\epsilon k}^{n+j} q_{\delta i}^{n+\kappa} + q_{\delta k}^{n+j} q_{\epsilon i}^{n+\kappa} = -2q_{\epsilon\delta}^{n+1} \delta_j^i \quad \forall \epsilon, \delta, j, k, i$$

Consider the map

$$(10.19) \quad \begin{aligned} \ker II_v &\rightarrow \text{End}(T/P_{sing}) \\ w^\epsilon e_\epsilon &\mapsto w^\epsilon q_{j\epsilon}^{n+\kappa} (e_j)^* \otimes e_k \end{aligned}$$

By (10.18), the fundamental lemma of Clifford algebras applies. \square

§11 RECOGNIZING UNIRULED VARIETIES

I now prove (0.10.2) stated in the introduction, in fact a more general result, but first we need a technical definition:

Fix $x \in X$ and let L denote an osculating linear space. Assume the Gauss map of X is nondegenerate Let $\pi : \mathcal{F} \rightarrow X$ denote the frame bundle. If L osculates at least to order $k+1$, then when one restricts the differential invariants $F_j \in \pi^*(S^j T_x^* X \otimes N_x X)$ to $S^j T_x L^* \otimes T_x L^\perp$, one obtains maps

$$(11.1.2) \quad R_j : S^j T_x L^* \otimes N \rightarrow T_x L \otimes T_x L^\perp \bmod \text{Image } R_{j-1}.$$

If these maps all have maximal rank for $2 \leq j \leq m$, we will say that *the maximal rank condition holds through level m* . The maximal rank condition is a pointwise genericity condition on the subspace of the space of tensors for F_2, \dots, F_m having the property that there is a k -plane in their common base locus.

11.1.3 Theorem [L7] (Expectation Theorem). *Let (n, a, k, m) be natural numbers satisfying $m \geq 3$ and*

$$a[\binom{k+m-1}{m-1} - k - 1] \geq k(n - k).$$

Let $X^n \subset \mathbb{A}^{n+a}$ or $X^n \subset \mathbb{P}^{n+a}$ be a open subset of a smooth (respectively analytic) submanifold of an affine or projective space having the properties that at each (resp. at a general) $x \in X$ there exists a k -dimensional linear space L_x , disjoint from the fiber of the Gauss map, osculating to order m and such that the maximal rank condition holds through level $m - 1$. Then L_x is locally contained in X .

If F denotes the fiber of the Gauss map at x and $\dim L_x \cap F = \lambda$, then the same conclusion holds as above with k, n respectively replaced by $k - \lambda, n - \lambda$. In fact, if the Gauss map is degenerate, one can replace L by the span of L and F .

11.1.4 Theorem [L7]. *Let $X^n \subset \mathbb{C}^{n+a}$ or $X^n \subset \mathbb{CP}^{n+a}$ be a open subset of a complex analytic submanifold of an affine or projective space, and let $x \in X$ be a general point. If $n \geq 4$ and a linear space L^{n-2} osculates to order four at x , then locally $L^{n-2} \subset X$.*

11.1.5 Proposition [L7]. *There exist analytic open subsets of varieties $X^n \subset \mathbb{A}^{n+a}$ and $X^n \subset \mathbb{P}^{n+a}$ having a line osculating to order $\frac{n-1}{a} + 2$ at every point that are not ruled. In fact over \mathbb{C} , every open subset of a variety has this property.*

Proof of 11.1.3. For notational simplicity, I only prove the case γ is nondegenerate. Let $(v_1, \dots, v_n) = (v_\xi, v_\rho)$ denote a basis of $T_x X$ adapted such that $T_x L = \{v_\xi\}$. The index ranges are $n + 1 \leq \mu, \nu \leq n + a$, $1 \leq \xi, \eta \leq k$, $k + 1 \leq \rho, \sigma \leq n$.

By hypothesis $F_j(v_{\xi_1}, \dots, v_{\xi_j}) = 0$ for all $1 \leq j \leq m$. The coefficients of F_{i+1} in this range are given by the formula ([L6], 2.20) which simplifies under our hypotheses to

$$(11.1.6) \quad r_{\xi_1, \dots, \xi_i, \rho}^\mu \omega_0^\rho = \mathfrak{S}_{\xi_1, \dots, \xi_i} r_{\xi_1, \dots, \xi_{i-1}, \sigma}^\mu \omega_{\xi_i}^\sigma.$$

(11.1.6) expresses the forms ω_ξ^σ in terms of the forms ω_0^ρ . Geometrically, say we were in the case of a unique linear space L at each point. Then we would have a map:

$$(11.1.7) \quad \begin{aligned} l : X &\rightarrow G \\ x &\mapsto L_x \end{aligned}$$

where G denotes the appropriate Grassmannian of k planes. In this case the forms ω_ξ^σ correspond to a spanning set of the cotangent space of the image. In the general case one still has such a map, only from the bundle over X whose fibers parametrize the space of L 's through a point. We wish to show the mapping is constant along tangent directions to L , that is the pullback of the forms ω_ξ^σ are zero when restricted to $T_x L$.

In terms of tensors, for each element of $S^i T_x L^* \otimes N$ we obtain a (possibly zero) element of $T_x L^\perp$ that is identified with an element of $T_x L^* \otimes T_x L^\perp$. Suppressing the element of $T_x L^\perp$ from the notation, we obtain (11.1.2).

We see that if these maps are all of largest possible rank, then at level i we will have filled a

$$(11.1.8) \quad a[\binom{k+1}{2} + \binom{k+2}{3} + \dots + \binom{k+(i-2)}{i-1}] = a[\binom{k+i-1}{i-1} - i - 1]$$

dimensional subspace of $T_x L^* \otimes T_x L^\perp$, which is of dimension $(n - k)k$.

Assuming the maximal rank condition holds through level $m - 1$ and $a[(\binom{k+m-1}{m-1}) - k - 1] \geq k(n - k)$, we see that $\omega_\eta^\sigma \equiv 0 \pmod{\{\omega_0^\rho\}}$ $\forall \sigma, \eta$. Intuitively, by the remark above, this finishes the proof. In details, by induction, for $i > m$ we have

$$(11.1.9) \quad r_{\xi_1, \dots, \xi_i \xi_{i+1}}^\mu \omega_0^{\xi_{i+1}} + r_{\xi_1, \dots, \xi_i \rho}^\mu \omega_0^\rho = \mathfrak{S}_{\xi_1, \dots, \xi_i} r_{\xi_1, \dots, \xi_{i-1} \sigma}^\mu \omega_{\xi_i}^\sigma,$$

which implies $r_{\xi_1, \dots, \xi_i \xi_{i+1}}^\mu = 0$ for all i , and thus the linear space osculates to infinite order. \square

To prove theorems 1 and 5, one observes that if the maximal rank condition fails in these cases, the relevant invariants are forced to be zero anyway.

§12 QUADRICS CONTAINING X

In this section I derive the generalized Monge system described in (0.6).

X is locally the intersection of quadrics if $N_x^* X$ is spanned by the differentials of quadratic equations.

In order that N_x^* be spanned by differentials of quadratic polynomials, it is necessary that

$$(12.1.k) \quad \{dP_x | P \in \ker \mathbb{FF}_{v_2(X)}^k\} = N_x^*$$

for all k . (We suppress reference to the base point x in what follows.) For $k \leq 2$, (12.1.k) automatically holds; for $k = 3$ (12.1.3) will hold if and only if

$$(12.2) \quad \mathbb{FF}^{3\mu} = 3a_{\nu\gamma}^\mu \omega_0^\gamma \mathbb{FF}^{2\nu}$$

for some constants $a_{\nu\gamma}^\mu \in \mathbb{C}$. Notice that if $r_{\alpha\beta\gamma}^\mu = \mathfrak{S}_{\alpha\beta\gamma} a_{\nu\gamma}^\mu q_{\alpha\beta}^\nu$ in some frame, it holds in any choice of frame (with different constants $a_{\nu\gamma}^\mu$), so the expression (12.2) has intrinsic meaning. If (12.2) holds, then

$$(12.3) \quad \ker \mathbb{FF}_{v_2(X)}^3 = \{x^\mu x^0 - q_{\alpha\beta}^\mu x^\alpha x^\beta - a_{\nu\beta}^\mu x^\nu x^\beta, x^\mu x^\nu\}.$$

Continuing in the same fashion, we uncover the following conditions:

$$(12.4) \quad \begin{aligned} \mathbb{FF}^{3\mu} &= 3a_{\nu\gamma}^\mu \omega_0^\gamma \mathbb{FF}^{2\nu} \\ \mathbb{FF}^{4\mu} &= 4a_{\nu\alpha}^\mu \omega_0^\alpha \mathbb{FF}^{3\nu} + 3b_{\nu\tau}^\mu \mathbb{FF}^{2\nu} \mathbb{FF}^{2\tau} \\ \mathbb{FF}^{5\mu} &= 5a_{\nu\gamma}^\mu \omega_0^\gamma \mathbb{FF}^{4\nu} + 10b_{\nu\tau}^\mu \mathbb{FF}^{3\nu} \mathbb{FF}^{2\tau} \end{aligned}$$

where $a_{\nu\alpha}^\mu, b_{\nu\tau}^\mu = b_{\tau\nu}^\mu \in \mathbb{C}$. Moreover, if there are no linear syzygies among the quadrics in $|II|$, as explained in §14, then $\mathbb{FF}^6 = 0$ and thus N_x^* is spanned by the differentials of quadrics and these quadrics are smooth along X so they generate $I(X)$. In this case, we will call (12.4) the *generalized Monge system for quadrics*.

In summary:

12.5 Theorem [L6]. *Let $X \subset \mathbb{P}V$ be a variety and $x \in X$ a general point. Assume $III_{X_x} = 0$ and that there are no linear syzygies in $|II|_x$. Then*

$$(12.6) \quad \begin{aligned} \dim \{ \text{quadrics osculating to order three at } x \} &\leq a + \binom{a+1}{2} - 1 \\ \dim \{ \text{quadrics osculating to order four at } x \} &\leq a - 1. \end{aligned}$$

If the generalized Monge system (12.4) holds, then

$$I_2 = \ker \mathbb{FF}_{v_2(X)_x}^4$$

Equality occurs in the first (respectively second) line of (12.6) if and only if the first (resp. second) line of (12.4) holds at x . If the generalized Monge system does not hold, then I_X is not generated by quadrics.

If one assumes appropriate genericity conditions, there exist analogous Monge equations for I_d of order $2d + 1$ in small codimension, see [L6].

§13 RECOGNIZING HOMOGENEOUS VARIETIES

Before studying homogeneous varieties, let's consider a general question:

If one fixes the dimension and codimension of X , there is an integer k_0 such that if F_2, \dots, F_{k_0} satisfy some mild genericity hypotheses, then all F_l 's are determined by the F_2, \dots, F_{k_0} 's and their derivatives. Since X is analytic, this means that X is entirely determined by k_0 derivatives on any open set. For a hypersurface, with $n \geq 2$, Jensen and Musso [JM] proved that $k_0 = 3$ (the case $n = 2$ is due to Cartan and Fubini). For a plane curve, $k_0 = 6$. In general, the function $k_0(n, a)$ appears to be unknown.

13.1 Problem. What, if any, values of a have $k_0(n, a) = 2$?

Note that the Codazzi equation in Euclidean geometry implies that if $III^{\mathbb{E}} = 0$, then $k_0^{\mathbb{E}} = 2$.

13.2 Locally uniform varieties.

Now let's specialize our question to the case of *locally uniform* varieties.

13.2.1 Definition. A variety $X \subset \mathbb{P}V$ is *locally uniform to order k* in the neighborhood of a point $x \in X$ if there exists a local framing (section of \mathcal{F}^1) in which the coefficients of F_l are constant for $2 \leq l \leq k$.

For example, all hypersurfaces are locally uniform to order two in neighborhoods of general points.

13.2.3 Remark. R. Bryant has proven that varieties that are locally uniform to order ∞ are in fact locally homogeneous (personal communication).

13.2.2 Questions. How does (13.1) simplify for locally uniform varieties? Are locally uniform varieties always rational (assuming $a \neq 1$ and perhaps some additional conditions)? How can one characterize the varieties of codimension two that are locally uniform?

Recall that the coefficients of F_3 are given by:

$$r_{\alpha\beta\gamma}^\mu \omega_0^\gamma = -dq_{\alpha\beta}^\mu - q_{\alpha\beta}^\mu \omega_0^0 - q_{\alpha\beta}^\nu \omega_\mu^\nu + q_{\alpha\gamma}^\mu \omega_\beta^\gamma + q_{\gamma\beta}^\mu \omega_\alpha^\gamma$$

13.2.3 Exercise. We can always take a local framing such that $\omega_0^0 = 0$.

On any section of \mathcal{F} we have $\omega_\nu^\mu = a_{\nu\alpha}^\mu \omega_0^\alpha$, $\omega_\beta^\alpha = b_{\beta\gamma}^\alpha \omega_0^\gamma$ for some functions $a_{\nu\alpha}^\mu, b_{\beta\gamma}^\alpha$. Thus, assuming $dq_{\alpha\beta}^\mu = 0$, i.e. local uniformity to order two, we obtain restrictions on F_3 . Namely, at each point, let

$$(13.2.3) \quad A \in N \otimes N^* \otimes T^* \quad B \in T^* \otimes T \otimes T^*$$

then

$$(13.2.4) \quad F_3 \in A \cdot II + S(B \cdot II)$$

where $A \cdot II, B \cdot II$ are the natural contractions and S is symmetrization in *two* factors.

F_3 is symmetric in all three factors which places significant restrictions on the admissible A 's and B 's. Moreover, F_3 can be modified by $T^* \circ II + \langle II, T^* \otimes N, II \rangle$ so one might hope that using all these conditions to show F_3 must be zero in certain situations, or at least severely limited.

13.2.6 Problem. Give a clean cohomological description of the restrictions described above.

In the example below, these conditions are *not* enough to conclude $F_3 = 0$, higher order considerations must be taken into account.

Further differentiation places systems of partial differential equations on the functions $a_{\nu\alpha}^\mu, b_{\beta\gamma}^\alpha$, and it is these overdetermined systems of pde that finally allow one to conclude $F_3 = 0$ in the calculations below.

13.3 Homogeneous varieties.

Homogeneous varieties usually have rather special differential invariants. For example, II of a homogeneous variety $X = G/P$ must be invariant under the action of the semi-simple part of P on $S^2 T^* \otimes N$.

We have seen in the introduction that even when II is invariant under a large group, it may not determine X (e.g. the adjoint varieties).

13.4 Idea of the proofs.

The idea of the proofs is as follows: given any variety $X \subset \mathbb{P}V$, consider the first order adapted frame bundle, $\pi : \mathcal{F}_X^1 \rightarrow X$. Each $f \in \mathcal{F}^1$ determines a splitting of the flag $\hat{X} \subset \hat{T} \subset V$ which we denote $\hat{x} + T + N$.

Write the pullback of the Maurer-Cartan form of $GL(V)$ to \mathcal{F}^1 as

$$\Omega = \begin{pmatrix} \omega_0^0 & \omega_\beta^0 & \omega_\nu^0 \\ \omega_0^\alpha & \omega_\beta^\alpha & \omega_\nu^\alpha \\ 0 & \omega_\beta^\mu & \omega_\nu^\mu \end{pmatrix}$$

with index ranges $1 \leq \alpha, \beta \leq \dim X$, $\dim X + 1 \leq \mu, \nu \leq \dim \mathbb{P}V$.

If $X = G/P$, one can reduce \mathcal{F}^1 until it is isomorphic to G (with fiber isomorphic to P). In that case one obtains the Maurer-Cartan form symbolically as:

$$\Omega_G = \begin{pmatrix} \omega_0^0 & \omega_\beta^0 & 0 \\ \omega_0^\alpha & \omega_\beta^\alpha = \rho_T(\mathfrak{h}) & \omega_\nu^\alpha = A_2(\omega_\beta^0) \\ 0 & \omega_\beta^\mu = A_1(\omega_0^\alpha) & \omega_\nu^\mu = \rho_N(\mathfrak{h}) \end{pmatrix}$$

where H is the semi-simple part of P , $T = T_x X$, $N = N_x X$ are H -modules with representations ρ_T , ρ_N , and A_1, A_2 are H -equivariant maps. The zero in the upper right hand block indicates that any infinitesimal change in the splitting satisfies the “transversality” condition that $dN \subseteq \{T + N\}$. The dependence of the ω_ν^α block on the forms ω_β^0 indicates that if one changes the choice of T , there is a corresponding change in choice of N mandated.

If X is a variety with the same second fundamental form as G/P , by restricting bases we can reduce \mathcal{F}_X^1 to a bundle \mathcal{F}_X^2 where the pullback of the the Maurer-Cartan form looks like:

$$\Omega = \begin{pmatrix} \omega_0^0 & \omega_\beta^0 & \omega_\nu^0 \\ \omega_0^\alpha & \omega_\beta^\alpha = \rho_T(\mathfrak{h}) + w_1 & \omega_\nu^\alpha \\ 0 & \omega_\beta^\mu & \omega_\nu^\mu = \rho_N(\mathfrak{h}) + w_2 \end{pmatrix}$$

where w_1, w_2 are linear combinations of the other forms appearing in the Maurer-Cartan form. The proofs proceed by showing that there are reductions of \mathcal{F}_X^2 to G .

In practice, the rigidity proofs proceed by showing the invariants $F_k \in \pi^*(S^k T^* X \otimes N X)$ are zero for $k > 2$.

13.5 Proof of the Grassmannian case.

$z = 0$ because $|II|^{(1)} = 0$.

Let V have basis $\{e_0, e_{1j}, e_{2j}, e_{jk}\}$, where $3 \leq j, k, l \leq n+2$, $\{\alpha\} = \{1j, 2j\}$. Normalize such that $II = (\omega_0^{1j} \omega_0^{2k} - \omega_0^{1k} \omega_0^{2j}) \otimes e_{jk}$, $j < k$. Note that the forms $\omega_{1j}^{1i}, \omega_{2j}^{2i}, \omega_{1i}^{1i}, \omega_{2i}^{2i}, \omega_{1i}^{2i}, \omega_{2i}^{1i}$ are all independent and independent of the semi-basic forms because they represent infinitesimal motions that preserve our normalization of II . We have

$$(13.5.1) \quad r_{(1k)(1l)\beta}^{ij} = 0 \quad \forall i, j, k, l \text{ distinct and } \forall \beta$$

$$(13.5.2) \quad r_{(2k)(2l)\beta}^{ij} = 0 \quad \forall i, j, k, l \text{ distinct and } \forall \beta$$

(these equations imply that the refined third fundamental form is zero). From now on, assume all indices are distinct. Using (13.5.1), (13.5.2), we have

$$(13.5.3) \quad \begin{aligned} r_{(1i)(1k)(1i)}^{(ij)} \omega_0^{1i} + r_{(1i)(1k)(1j)}^{(ij)} \omega_0^{1j} + r_{(1i)(1k)(2i)}^{(ij)} \omega_0^{2i} + r_{(1i)(1k)(2j)}^{(ij)} \omega_0^{2j} + r_{(1i)(1k)(2l)}^{(ij)} \omega_0^{2l} \\ = \omega_{(1k)}^{(2j)} \end{aligned}$$

The right hand side of (13.5.3) is independent of i , so comparing with the same expression using m instead of i , (here we use that $n \geq 4$) we obtain:

$$(13.5.4) \quad r_{(1i)(1k)(1i)}^{(ij)} = 0$$

$$(13.5.5) \quad r_{(1i)(1k)(1j)}^{(ij)} = r_{(1m)(1k)(1j)}^{(mj)}$$

$$(13.5.6) \quad r_{(1i)(1k)(2i)}^{(ij)} = r_{(1m)(1k)(2i)}^{(mj)}$$

$$(13.5.7) \quad r_{(1i)(1k)(2j)}^{(ij)} = r_{(1m)(1k)(2j)}^{(mj)}$$

$$(13.5.8) \quad r_{(1i)(1k)(2l)}^{(ij)} = r_{(1m)(1k)(2l)}^{(mj)}.$$

Now

$$\begin{aligned}\Delta r_{(1i)(1k)(2l)}^{(ij)} &= g_{(kl)}^{(2j)} \\ \Delta r_{(1i)(1k)(2j)}^{(ij)} &= g_{(jk)}^{(2j)} + g_{(1k)}^0.\end{aligned}$$

Using these equations and the corresponding equations with the role of 1 and 2 reversed, we reduce to frames where $r_{(1i)(1k)(2l)}^{(ij)} = 0$, $r_{(1i)(1k)(2j)}^{(ij)} = 0$, $r_{(2i)(2k)(1l)}^{(ij)} = 0$, $r_{(2i)(2k)(1j)}^{(ij)} = 0$. In these frames, $\omega_{2j}^{1k}, \omega_{1j}^{2k} = 0$ hence

$$(13.5.9) \quad 0 = r_{(1i)(1i)\beta}^{(ij)} \omega_0^\beta = -2\omega_{1i}^{2j}$$

and similarly with the role of 1 and 2 reversed. Thus the only nonzero terms left in F_3 are $r_{(1i)(1j)(1k)}^{(ij)}, r_{(2i)(2j)(2k)}^{(ij)}$. Consider

$$(13.5.10) \quad r_{(1i)(1j)(1i)\beta}^{(ij)} \omega_0^\beta = 2r_{(1i)(1j)(1k)}^{(ij)} \omega_{1i}^{1k}$$

Both sides of (13.5.10) must be zero because the forms ω_{1i}^{1k} are all independent and independent of the semi-basic forms. The analogous equation holds with 2's. Hence we see $F_3 = 0$.

To have a nonzero coefficient of F_4 , $r_{\alpha\beta\gamma\delta}^{(ij)}$, in the lower indicies there must be two 1's and two 2's, and at least two of the k -indices must be i or j . Consider

$$(13.5.11) \quad r_{(1i)(1k)(2l)(2j)}^{(ij)} \omega_0^{2j} = \omega_{kl}^{2j}$$

$$(13.5.12) \quad r_{(1i)(1i)(2l)(2j)}^{(ij)} \omega_0^{2j} = 2\omega_{il}^{2j}.$$

Since the right hand side of (13.5.11) is independent of i , we conclude (after switching the roles of i and j) that $r_{(1i)(1k)(2l)(2j)}^{(ij)}$ is independent of i, j (with neither k, l equal to i or j , but $k = l$ is possible). Using

$$\Delta r_{(1i)(1k)(2l)(2j)}^{(ij)} = g_{kl}^0$$

we normalize all these terms to zero. This implies $\omega_{il}^{2j} = 0$ and thus $r_{(1i)(1i)(2l)(2j)}^{(ij)} = 0$ for all i, j, l distinct as well, and similarly with the role of 1 and 2 reversed. Thus the remaining nonzero terms in F_4 are $r_{(1i)(1i)(2j)(2j)}^{(ij)}, r_{(1i)(2i)(1j)(2j)}^{(ij)}$. Consider

$$\begin{aligned}(13.5.13) \quad & r_{(1i)(2j)(1k)(2l)(1i)}^{(ij)} \omega_0^{1i} + r_{(1i)(2j)(1k)(2l)(1j)}^{(ij)} \omega_0^{1j} \\ & + r_{(1i)(2j)(1k)(2l)(2i)}^{(ij)} \omega_0^{2i} + r_{(1i)(2j)(1k)(2l)(2j)}^{(ij)} \omega_0^{2j} \\ & = -\omega_{kl}^0.\end{aligned}$$

Since the right hand side of (13.5.13) is independent of i, j , we conclude $\omega_{kl}^0 = 0$ and hence the left hand side is zero as well. Now it is easy to see the rest of the terms in F_5 are zero and all higher forms are zero. \square

13.6 Problem. Determine rigidity for the case $G(2, 5)$.

§14 COMPLETE INTERSECTIONS

While the other pathologies we have so far studied could be calculated locally, failing to be a complete intersection is a global issue. For example at any $x \in X_{sm}$, X is locally a complete intersection. To attempt to recognize whether or not a variety is a complete intersection from computable local information would be futile. In this section I will discuss computable local conditions that insure X is a complete intersection.

Let $V = \mathbb{C}^{n+a+1}$ and let $X^n \subset \mathbb{P}V = \mathbb{CP}^{n+a}$ be a variety of dimension n . Let X_{sm} denote the smooth points of X . Let $I_X \subset S^\bullet V^*$ denote the ideal of X and let $I_{X,d} = I_d = S^d V^* \cap I_X$ denote the d -th graded piece of I_X . Fixing a smooth point $x \in X$, there is a distinguished subspace of I_d , namely the hypersurfaces of degree d that are singular at x , i.e. $P \in I_d$ such that $(dP)_x = 0$, where dP denotes the exterior derivative of the polynomial P .

The following definition is due to Lvovsky [Lv]:

14.1 Definition [L6]. . Let $X \subset \mathbb{P}V$ be a variety. Let $P \in I_d$ and let $Z = Z_P \subset \mathbb{P}V$ be the corresponding hypersurface. We will say Z *trivially contains* X if $P = l^1 P_1 + \dots + l^m P_m$ with $P_1, \dots, P_m \in I_{d-1}$ and $l^1, \dots, l^m \in V^*$, and otherwise that Z *essentially contains* X .

[L6, 1.1] stated in the introduction generalizes to the following statement:

14.2 Proposition [L6,1.6], A local characterization of complete intersections. *Let $X \subset \mathbb{P}V$ be a variety. The following are equivalent:*

1. *X is a complete intersection.*
2. *Every hypersurface essentially containing X is smooth at all $x \in X_{sm}$.*
3. *Let $x \in X_{sm}$. Every hypersurface essentially containing X is smooth at x .*

(14.2) localises the study of complete intersections to a point, and further, filters the conormal bundle at that point to enable us to study one degree at a time. Unfortunately, to determine if a hypersurface essentially contains X , one might need to take an arbitrarily high number of derivatives. To have computable conditions, we will work with osculating hypersurfaces rather than the hypersurfaces containing X . The advantage will be that we will only need to study a fixed number of derivatives for each fixed degree of hypersurface; the disadvantage is that we will only obtain sufficient conditions to be a complete intersection.

By [L4, 3.16, 3.17] stated in the introduction, we see that at best one could prove there are no singular hypersurfaces of degree d osculating to order $2d + 2$ at x ; and that the first restrictions one could hope for are at order $d + 1$.

I now specialize to the case $d = 2$.

Looking at (4.6.2), we see that $\ker \mathbb{FF}_{v_2(X),x}^3$ is as small as possible if there are no *linear syzygies* among the quadrics in $II_{X,x}$.

(If $A \subset S^2 T^*$, a linear syzygy among the quadrics in A is a relation of the form $l_j \circ Q^j = 0$, where $l_j \in T^*$ and $Q^j \in A$. More invariantly, consider the symmetrization map $\mathcal{S} : T^* \otimes S^2 T^* \rightarrow S^3 T^*$ and its restriction to A , $\mathcal{S}|_A : T^* \otimes A \rightarrow S^3 T^*$. Let $A^{[1]} = \ker(\mathcal{S}|_A)$. Then $A^{[1]}$ is the space of linear syzygies of A .)

We have the following linear algebra lemma:

14.3 Lemma [L4, 6.19]. *Let $A^p \subset S^2 T^*$ be an p -dimensional system of quadrics on an n dimensional vector space. Say there is a linear syzygy*

$$l^1 Q_1 + \dots + l^p Q_p = 0$$

where both $l^i \in T^*$ and $Q_i \in A$ are independent sets of vectors. Then $\forall Q \in A$,

$$\text{rank } Q \leq 2(p-1).$$

For the proof, see [L6]. If one now compares [L4, 6.19] with the rank restriction theorem, one sees that if $a < \frac{n-(b+1)+3}{3}$ then there are no linear syzygies in $|II|$. Combined with the generalized Monge system, we obtain:

14.4 Theorem [L6, 6.26]. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and $x \in X$ a general point. Let $b = \dim X_{\text{sing}}$. (Set $b = -1$ if X is smooth.) If $a < \frac{n-(b+1)+3}{3}$ then*

$$(12.5) \quad \begin{aligned} \dim \{ \text{quadrics osculating to order three at } x \} &\leq a + \binom{a+1}{2} - 1 \\ \dim \{ \text{quadrics osculating to order four at } x \} &\leq a - 1. \end{aligned}$$

Equality occurs in the first (respectively second) line of (14.5) if and only if the generalized Monge system holds to order three (respectively four) at x . If the generalized Monge system holds, then X is a complete intersection of the $a-1$ dimensional family of quadrics osculating to order four.

[L4, 6.28] stated in the introduction follows immediately in the following stronger form:

14.6 Corollary [L6, 6.28]. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and $x \in X$ a general point. Let $b = \dim X_{\text{sing}}$. (Set $b = -1$ if X is smooth.) If $a < \frac{n-(b+1)+3}{3}$ then any quadric osculating to order four at x is smooth at x and any quadric osculating to order five at x contains X .*

While the higher order Monge equations are more complicated to write down, in principle they are no more difficult to understand. Thus the problem of determining the subbundle of the conormal bundle consisting of trivially containing hypersurfaces is in principle resolved. The degeneracy conditions on the differential invariants of X , while more complicated to write down, is also, in principle resolved. The first condition corresponds to the existence of (excess) linear syzygies. The first real problem in attempting to generalize (14.4) is that for polynomials of higher degrees, the presence of linear syzygies is not a serious pathology, in particular, it has little relation to the “cone locus” (maximal multiplicity locus) of the polynomials. Moreover, while there are indications that there may be higher order rank restriction theorems, none have yet been proven, and any I would be willing to conjecture would relate to restricting the cone locus.

§15 ERRATA AND CLARIFICATIONS

1. In [L3] p 315, it says that the Grassmannians $G(2, m)$ will have degenerate γ_k . It should say the Segre's $\text{Seg}(\mathbb{P}^l \times \mathbb{P}^m)$ will have degenerate γ_k .

2. In [L6] I give an example of a variety with $F_3 = 0$ but $F_4 \neq 0$ at a point, but I do not show that that point is a general point. An example where the phenomena does occur at general points is a cone over a curve in \mathbb{P}^2 .

3. In [L5] 12.1, I state that when X is smooth and $\tau(X)$ is degenerate, the fibers of the Gauss map of $\tau(X)$ are of dimension at least $\delta_\tau + 2$, but what is proven is that $\tau(X)$ has dual defect at least $\delta_\tau + 2$. The statement as announced is false. Moreover, the observation of the defect of $\tau(X)^*$ was already made by Zak [Z].

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